Mathematical Knowledge for Teaching using Technology

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Abstract

The focus of this PhD study is teachers' knowledge and how it is involved in interacting with technology to produce the mathematical knowledge made available in the classroom. Contrasting connectionist and transmissionist teachers' use of technology provides a means of making such knowledge visible, allowing an exploration of the nature and content of mathematical knowledge for teaching using technology. In addition, this study examines how and to what extent the mathematical knowledge made available through a teacher's interaction with technology is distributed across the teacher and technology.

The first, quantitative phase of the project surveyed English secondary mathematics teachers' use of technology (n=183). Using Rasch analysis to construct a transmissionist measure of self-reported pedagogic practice, a surprising association is found between frequent use of teacher-centred software and a more connectionist orientation. The survey data also suggests that 'teacher-centred' practices involving ICT may instead be construed as 'dominant' practices, since they are most frequently occurring across all teachers.

In the second, qualitative phase of the project, two connectionist and two transmissionist teachers were selected as case studies on the basis of their responses to the survey instrument. Data collection involved a semi-structured interview based around a GeoGebra file on circle theorems, two classroom observations and post-observation interviews. Data analysis using the TPACK framework suggests the nature of mathematical knowledge for teaching using technology as abstract, mathematical knowledge and yet simultaneously as mathematical knowledge *situated* in the context of teaching using technology. Using the Knowledge Quartet, a conceptualisation of the content of mathematical knowledge for teaching using technology in relation to the topic of circle theorems is developed, demonstrating the highly complex nature of such knowledge. Ameliorating this complexity, this study provides indications of how a distributed view of cognition might offer potential strategies for facilitating teacher interaction with technology.

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Chapter 1 - Introduction

This thesis brings together the fields of mathematics teacher knowledge and technology integration into classroom practice to explore teachers' knowledge in relation to using technology to teach mathematics. Using survey data, classroom observations and interviews, this thesis describes the nature and content of individual teachers' knowledge and how it is involved in interacting with technology to produce mathematical knowledge made available in the classroom. This contributes towards a broader aim to develop a deeper understanding of both how and why mathematics teachers use technology in their classroom practice.

This introductory chapter begins by providing some contextual background: describing the author's personal motivations for undertaking the study as well as setting the study in a wider context. The following sections then describe and justify the theoretical framework for understanding individual teachers' knowledge and how it is involved in interacting with technology, which underpins this study. Key terms, such as 'technology' and 'knowledge' are also defined. Returning to the context of the study, two main research questions and two subsidiary questions are identified. The chapter concludes with a brief description of the structure of the rest of the thesis.

1.1 Setting the scene: the personal and wider context for the study

The original impetus for this United Kingdom (UK) Economic and Social Research Council (ESRC) funded PhD research project came from the difficulties the author experienced, as a secondary school mathematics teacher, in trying to integrate digital technology into her own classroom practice, despite viewing herself as a competent and confident user of such technology. She explored her own use of digital technology in teaching, specifically the development of teaching techniques for using dynamic geometry software, through an empirically-based post-graduate Master of Arts (MA) dissertation (Bretscher, 2007). This fuelled an interest in developing a deeper understanding of the complexities of integrating digital technologies experienced by teachers more widely and the apparent deficit between aspirations for technology use in schools and the classroom reality of technology use. These issues were explored further through a post-graduate Master in Research Methods (MRes) study (Bretscher, 2009) employing frameworks drawn from the literature on digital technology in mathematics education to analyse teachers' classroom practice using digital technology. The MRes study informed the PhD research project, providing a first pilot for the qualitative phase of data collection. In addition, attempting to develop a deeper understanding of teachers' classroom practice using digital technology led to the progressive focusing of the PhD research project on teachers' mathematical knowledge for teaching using technology. This focus feeds into the broader aim of the study to develop a deeper understanding of both how and why mathematics teachers use technology in their classroom practice.

1.1.1 A deficit in mathematics teachers' use of digital technologies?

A deficit in mathematics teachers' use of digital technologies appears to exist in two senses. Firstly, mathematics teachers appear to make only occasional use of digital technologies in their teaching. Secondly, when teachers *do* make use of digital technologies, the potential of these technologies to enhance pupils' mathematical experience in the classroom is rarely realised. This apparent dual deficit is particularly significant in the context of unprecedented spending by governments around the world on initiatives to develop educational technology (Selwyn, 2000) and the emphasis placed on using Information and Communication Technologies (ICT) in previous UK National Curricula (e.g. 2007) for mathematics and the inclusion of digital technologies in mathematics curricula more globally (Wong, 2003).

There seems to be unequivocal evidence for the deficit in terms of the frequency with which mathematics teachers use ICT. The 2007 Trends in International Mathematics and Science Study (TIMSS: Mullis, Martin, & Foy, 2008) reports that it was rare for computers to be used for any activity as often as in half the mathematics lessons, even in countries with relatively high availability. The Eurydice report (2011) on ICT use in European schools states that only a minority of mathematics teachers have successfully embedded digital technologies into their lessons and that computers are most often used for skill practice in mathematics. In the UK, the *Impact of ICT on Pupil Learning and Attainment* report (Harrison et al., 2003) stated that 67% of pupils at Key Stage 3 never or hardly ever used ICT in their mathematics lessons. In addition, Selwyn (2008) noted that despite the previous years of unprecedented investment by the then Labour government, directing over £5 billion of funding towards educational ICT during the 1997 to 2007 period, the UK Office for Standards in Education (Ofsted, 2008) reported that opportunities for pupils to use ICT to solve or explore mathematical problems had markedly decreased. Four years later, Ofsted (2012) reported there had been little change in teachers' use of ICT, concluding that the potential of ICT to enhance learning in mathematics continued to be underdeveloped.

The conclusions of the Ofsted report (2012) provide an indicator of the deficit in terms of the extent to which the potential of digital technologies remains unfulfilled. That is, where mathematics teachers *do* make use of digital technologies, the potential of these technologies is rarely realised. Lagrange and Erdogan (2008) refer to this as a qualitative gap in ICT use, citing *inter alia* Ruthven and Hennessy's (2002) study of mathematics teachers in England as evidence. Lagrange & Erdogan (2008, p.66) refer to the tendency of teachers to view the benefits of technology in terms of enabling "general 'pedagogical' aspirations rather than for its 'didactical' contribution to mathematics learning". In Ruthven and Hennessy's study, mathematics teachers articulated the benefits of technology as indirectly enhancing students' learning through increased pace and productivity and improved engagement rather than providing a direct means of enhancing mathematics pedagogy.

Hence, many mathematics teachers do integrate some types of digital technology into their classroom practice. Moss et al's (2007) survey on the introduction of interactive whiteboards (IWBs) in London schools shows that many teachers are using IWBs in most or every lesson. This was especially true in mathematics and science, where the majority did so (65% in mathematics). Moreover, in case study lessons, they observed mathematics teachers using presentation software, such as ActivStudio, as well as subject-specific software such as Geometer's Sketchpad (Key Curriculum Press, 2003). In the US, Becker, Ravitz and Wong (1999) found that drill-and-practice software was the type of software most often used by mathematics teachers. Descriptive statistics of teachers' technology use, based on data from this PhD study but published elsewhere (Bretscher, 2014), suggest that English secondary mathematics teachers do make frequent use of technology. Bretscher (2014) reports that IWBs were used in almost every lesson by 85% of the responding teachers ($n = 175$). Presentation-oriented software dominates IWB use, whilst the *MyMaths* web-site (Oxford University Press, 2012) offering pre-prepared lessons dominates teachers' use of computer suites as well as featuring prominently amongst software used with IWBs (Bretscher, 2014).

However, the digital technologies that are relatively widely implemented tend to be compatible with and facilitate whole-class instruction through enhanced presentation.

These are often perceived as teacher-centred (rather than student-centred) (e.g. Zevernbergen & Lerman, 2008) and viewed as maintaining or even encouraging existing 'traditional' pedagogies. These types of technology tend to be overlooked in the literature (e.g. Pierce & Stacey, 2010; Zbiek et al., 2007) or even dismissed as 'not really using ICT'. For example, Zbiek et al (2007) state that they do not include digital technologies whose primary purpose is as a presentation tool rather than as a problem-solving or mathematical tool in their review of research. The deficit in terms of the frequency of ICT use appears therefore only in relation to digital technologies valorised by the mathematics education research community (e.g. Kaput, 1992; Zbiek et al., 2007). This suggests that mathematics education research on technology integration has tended to focus on understanding a minority of practice and the reasons why using digital technologies, such as those identified by Zbiek et al (2007) as *cognitive technological tools* or Pierce and Stacey (2010) as mathematical analysis software, remains a minority practice. As a result, widespread practice involving ICT remains under-analysed and (potentially) this has also had the effect of presenting an overly pessimistic view of teachers' classroom use of ICT. Thus the deficit in teachers' use of digital technologies in general may be over-estimated: both in terms of the low frequency of use and of the extent to which the potential of digital technologies remains unrealised.

1.1.2 Defining technology

This study focuses on digital technologies as a means of understanding teachers' use of technology more generally. This section first defines the more general term 'technology' and then goes on to define what is meant by 'digital technologies' in relation to this study.

In this study, 'technology' is used to indicate an artefact (physical or virtual) that has been designed for use or has been appropriated for use in teaching mathematics. This definition follows from Ruthven's (2009) usage of 'resource' and, in a sense, is intended to be interchangeable with this term. Why then use the term technology instead of resource? Technology has tended to be used to indicate and abbreviate digital technologies, separating them from non-digital technologies, to mark them out as worthy of special consideration or as unique in some way. For example, both Kaput (1992) and Zbiek et al's (2007) chapters focusing on digital technology in subsequent editions of the *Handbook of Research on Mathematics Teaching and Learning* are entitled Technology in Mathematics Education. Instead, in this study, whilst digital technologies are the focus, they are viewed simply as a particular type of technology amongst others and not as something special or unique. Referring to both digital and non-digital alike using the term ‘technology’ makes this point more salient precisely because it is perhaps less usual to call non-digital technologies, such as textbooks, a ‘technology’. Mishra and Koehler (2006, p. 1023) state that ‘traditional’ technologies have become so well integrated into teachers’ classroom practice they are not even regarded as technologies. Using the term ‘technology’ in this study is intended to serve as a reminder that artefacts such as textbooks, non-interactive whiteboards and graph paper are also technologies of teaching. The term ‘resource’, as Ruthven (2009) suggests, follows the everyday language of teachers and therefore appears more easily applicable to both digital and non-digital technologies. For this reason, ‘resource’ does not supply a cognitive prompt to the author at least in the way that the term ‘technology’ does, to recall that ‘traditional’ technologies, particularly those that are non-digital, are nevertheless types of teaching technology.

In the previous section, the deficit in teachers’ use of digital technology was identified with digital technologies valorised by the mathematics education community, such as those singled out by Zbiek et al (2007) as *cognitive technological tools* (abbreviated to cognitive tools). However, limiting the definition of digital technology to those described as cognitive tools tends to omit the very types of hardware and software that teachers do manage to integrate into their classrooms. It was argued that this has prevented an analysis of widespread practice. For the purposes of this PhD study a broader definition of digital technology was therefore adopted. Following from the definition of ‘technology’ used in this study, ‘digital technologies’ is used to indicate a digital artefact (physical or virtual) that has been designed for use or has been appropriated for use in teaching mathematics. This definition is broader than Zbiek et al’s definition because it incorporates any digital artefact used or designed for use in teaching mathematics, including those that are not obviously “mathematical tools” for example. In particular, it includes teacher-centred digital technologies such as presentation-oriented software (e.g. PowerPoint, IWB software and the *MyMaths* website) that appear to be prominent in teachers’ use of ICT, but excludes non-digital technologies such as textbooks. As with any technology, how and why teachers make use of the technology in their teaching is a central research question. Viewing digital technologies simply as a particular type of technology is in keeping with the broader aim of this study to develop a deeper understanding of both how and why mathematics teachers use technology in their classroom practice. Similarly, Ruthven (2009) and Gueudet and Trouche (2009) focus on digital technologies as a means of understanding teachers' use of technology more generally.

1.2 A socio-cultural perspective on teachers' technology use

Whilst many believe that technology has the power to transform education, Cuban's (1986, 1993, 2001) studies of educational innovation over time suggest otherwise, predicting that where digital technologies are taken up on a large-scale they will not fundamentally change teaching practice. Cuban's studies and the discussion in Section 1.1.1 show that integrating technology into the classroom is far from straightforward. In particular, Cuban's studies indicate a range of social factors influence teachers' use of technology, such as cultural beliefs, the school and classroom context. Socio-cultural perspectives appear to offer the most promising approach towards understanding teachers' technology use because they account for social factors in the context of teaching and learning. In addition, socio-cultural perspectives acknowledge that the design of a piece of software or hardware does not determine its use in the classroom. Instead, teachers interpret technology in the process of integrating it into their classroom practice. Thus a teacher's interpretation of a piece of technology is an important factor in shaping its end use in the classroom. For example, teachers who adopt a student-centred approach to pedagogy may interpret and use teacher-centred technologies, such as presentation-oriented software (e.g. PowerPoint, IWB software and the *MyMaths* website), to facilitate such an approach. If this is the case, then the deficit in teachers' use of digital technologies discussed in section 1.1.1, in terms of the extent to which the potential of digital technologies remains unrealised, may be over-estimated. Nevertheless, socio-cultural perspectives also acknowledge that the design of a piece of software or hardware constrains teachers' interpretations of the technology. Socio-cultural perspectives are useful in this study because they view teachers' use of technology as a two-way process where the teacher is engaged in a participatory relationship with the technology.

Hence a socio-cultural perspective on teachers' use of resources was adopted for this study, informed by Remillard's (2005) description of "curriculum use as participation with the text”. This perspective was developed in relation to ‘curriculum materials’, specifically referring to printed, often published resources designed for use by teachers and students during instruction. The definition of technology used in this study is broader than that of Remillard’s usage of the term ‘curriculum materials’ and her use of the term ‘text’, referring to the components that comprise curriculum materials. However it is narrower than the definition of ‘resources’ used by Gueudet and Trouche (2009) who also align their approach with Remillard’s perspective. For example, Gueudet and Trouche include “discussions with teachers orally or online” in their definition of resources. Thus, although Gueudet and Trouche (2009) also focus on digital technologies as a means of understanding teachers’ use of technology more generally, they seek to generalise their approach to a wider range of resources. The definition adopted in this study was narrower because it is assumed that the nature of the participatory relationship between an individual teacher and technology is likely to be different to a participatory relationship between teachers. Remillard’s (2005) perspective is appropriate for addressing the broader aim of this PhD study and provides an over-arching theoretical framework for understanding teachers’ use of technology.

1.2.1 Applying Remillard’s perspective to teachers’ use of technology

Remillard’s (2005) perspective views teachers as being engaged in a participatory relationship with technology, resulting in the planned and enacted curriculum i.e. the mathematical knowledge made available in the classroom. Hence understanding teachers’ use of technology – the broader aim of this study – means investigating what factors influence the participatory relationship and how they are involved in producing the mathematical knowledge made available in the classroom. Underlying Remillard’s (2005, p221) perspective are Vygotskian notions of tool[1] use, wherein technological tools both shape and are shaped by human action through their constraints and affordances. Applying Remillard’s perspective to digital technology implies that, although the constraints and affordances inherent in digital technologies may help to shape its end use in the classroom, inevitably, the end user, in this case individual teachers, will also work to shape the technology. Thus the design and nature of hardware or software is an ingredient in, but does not determine, the way individual

[1] “Tools” is used here to indicate a wider range of (physical or virtual) artefacts appropriated for human use i.e. not limited to those used for teaching mathematics. See also the use of the term “tools” in Hutchins’ (1995) view of distributed cognition.

teachers interpret and make use of particular digital technologies in their classroom practice. For example, Ruthven's (with Hennessy & Deaney, 2008; 2009) research on mathematics teachers' use of technology and in particular his notion of *interpretative flexibility* is commensurate with the perspective described by Remillard. Similarly, Gueudet and Trouche's (2009) outline of the documentational approach, extending the widely influential instrumental approach to teachers' appropriation of technology, shares the same Vygotskian roots as Remillard's perspective.

In Remillard's (2005) perspective both the teacher and technology play an active role in a dynamic and interactive relationship to produce the mathematical knowledge made available in the classroom. As a result, both the characteristics of the individual teacher and the technology are identified as key factors influencing this relationship. In particular, Stein et al (2007) identify teachers' knowledge and beliefs about mathematics teaching and learning and their orientation[2] towards a specific technology as teacher characteristics that affect the participatory relationship with the technology. Stein et al (2007) identify content and sequencing, as well as the way this is structured and communicated to the teacher, as characteristics of curriculum materials that affect the participatory relationship. The way these characteristics map to technology more generally needs consideration. A textbook or set of curriculum materials might be expected to cover the range of topics included in the curriculum. Thus content and sequencing in relation to curriculum materials could refer to the range of curricula topics covered and the way they are sequenced. Considering technology more generally, an individual artefact (physical or virtual) might address a particular topic – rather than a range of topics – or even a particular concept within a topic. Nevertheless, the content and sequencing of material designed to address a given topic or concept, and the way this is structured and communicated to the teacher, can be considered as characteristics of technology that affect the participatory relationship. Remillard's (2005) perspective also recognises the impact of students and context in enabling or constraining teachers' participation with technology. In particular, Stein et al (2007) highlight contextual features, such as *time* available for planning and instruction, *local* (school and departmental) *cultures* and *teacher support* through professional development, that can constrain or enable teachers' interpretations of curriculum materials. Similarly, Ruthven (2009) describes *working environment* and *time economy* as two of five structuring

[2] Stein et al (2007) use the term 'orientation' to indicate a teacher's stance towards a type of technology in relation to teaching mathematics. In this study, the term orientation is taken to include teacher's confidence in using technology to teach mathematics.

factors of classroom practice in relation to technology and Gueudet and Trouche (2009) include institutional influences as part of their model of the documentational approach.

No study can take all factors into account, thus all studies are necessarily limited in scope to a greater or lesser extent. The focus of this study is on individual teachers' knowledge and how it is involved in interacting with technology to produce the mathematical knowledge made available in the classroom. Other individual characteristics of teachers, characteristics of technology, students and contextual features identified in Remillard's perspective as important factors affecting the participatory relationship are acknowledged and where possible these are taken into account.

1.2.2 Knowledge, beliefs and orientation

In the Effective Teachers of Numeracy study carried out in the UK, Askew et al (1997) found that highly effective teaching of primary school mathematics was associated with what they termed a *connectionist orientation*. Their use of orientation suggested an amalgam of beliefs and knowledge in relation to teaching and learning mathematics and, in this sense, is similar to Stein et al's (2007) use of the term orientation in relation to a teacher's stance towards technology. In this study, 'orientation' is used in a similar way to suggest an amalgam of beliefs and knowledge, but also to include teacher confidence in relation to teaching and learning mathematics. In particular, in Chapter 4, 'ICT orientation' is taken to include a teacher's confidence in relation to using ICT to teach mathematics because research suggests this is an important factor related to technology use (e.g. Zammit, 1992).

Askew et al's (1997) definition of a connectionist orientation centred mainly on what they termed 'beliefs' rather than on what they termed 'knowledge'. Similarly, Zbiek and Hollebrands' (2008) distinguish teacher 'conceptions' from knowledge about mathematics and technology (see Chapter 2). Askew et al's (1997) use of beliefs and Zbiek and Hollebrands' use of conceptions refer to teachers' global beliefs about mathematics as a discipline and their beliefs about the nature of teaching and learning mathematics (using technology). However, distinguishing knowledge from beliefs is difficult because they are closely related (Scheffler, 1965) and hence distinctions between them are "fuzzy" (Thompson, 1992). As a result, in this study, beliefs and conceptions in relation to teaching and learning mathematics (using technology) are taken to be part of teacher knowledge.

Pampaka et al (2012) summarise what they term a “consensus view” amongst the English mathematics education community, represented by Advisory Committee on Mathematics Education (ACME) and National Centre for Excellence in Teaching Mathematics (NCETM), that promotes an approach to mathematics teaching which includes:

- connecting teaching to students’ mathematical understandings; and

- connecting teaching and learning across topics, and between mathematics and other areas of knowledge.

Amongst similar professional communities in the US there is also a consensus in relation to these two statements, although the first of these is more usually associated with constructivist views (Cobb, 1994), whilst the latter is associated with teaching for conceptual understanding (Kilpatrick, Swafford & Findell, 2001). In this study, it is asserted that these statements underpin a connectionist orientation to teaching mathematics.

1.2.3 What is mathematical knowledge for teaching using technology in terms of this study?

For the purposes of this study, borrowing from Shulman (1986, p.13), mathematical knowledge for teaching using technology is assumed not only to be a matter of knowing how – being competent in teaching mathematics using technology - but also of knowing what and why. That is, although much of teachers’ knowledge may be tacit (Ruthven, 2014, p. 390), at least some of their know-how is underpinned by articulated knowledge that provides for “a rational, reasoned approach to decision-making” (Rowland et al, 2005, p.260) in relation to teaching mathematics using technology. In other words, mathematical knowledge for teaching using technology, as defined in this study, is when know-how or knowledge-in-action is underpinned by and coincides with the teacher’s articulated knowledge. This intersection between articulated knowledge and knowledge-in-action (i.e. articulated knowledge-in-action) is important because it is this type of knowledge that initial teacher education (ITE) or continuing professional development (CPD) programmes focus on developing.

It is now possible to turn to the issue of how can we know about ‘knowledge’. Clearly, it is not possible to observe an individual teacher’s knowledge directly. Written assessments, such as those employed in Hill et al (2005), can provide a valid indicator of mathematical knowledge for teaching. However, a distinction must be acknowledged between what we say we do, articulated knowledge, and what we do, know-how. This relates to Argyris and Schon’s (1974, pp. 6-10) definition of “espoused theory” (theory to which we give our allegiance) and “theory-in-use” (theory which governs actions). That is, we cannot always do as we say (Adler, 2001): it is possible to infer knowledge from a teacher’s responses to a written assessment or interview that is not realised in the actuality of their classroom practice. Similarly, we cannot always say what we do (Adler, 2001). It is equally possible to infer knowledge from a teacher’s performance in the classroom that is not articulated during an interview or written assessment, e.g. Hodgen (2011) and Thwaites et al (2010). When articulation does coincide with performance, i.e. when a teacher appears to do as they have said or appears able to say what they have done, then this may provide a more reliable basis for inference about their mathematical knowledge for teaching using technology. In analysing individual teacher’s knowledge, it should be stated that the purpose of this study is to make inferences about the nature and content of teachers’ knowledge in general; it is not to critique the knowledge of individual teachers.

1.3 Understanding individual teachers’ knowledge in participation with technology

This section sets out a theoretical framework for understanding how an individual teacher’s knowledge is involved in the participatory relationship (Remillard, 2005) with technology to produce the mathematical knowledge made available in the classroom. Arguably, the mathematical knowledge made available in the classroom through the interaction of the teacher with technology has a more direct impact on the quality of pupils’ mathematical experience in the classroom than the knowledge of the individual teacher. The teacher in combination with technology might interact to augment or impede mathematical knowledge being made available in the classroom. That is, the mathematical knowledge made available in the classroom through a teacher’s interaction with technology might be greater or less than the sum of knowledge of the individual constituents of the teacher-technology combination.

1.3.1 Hutchins' view of distributed cognition

Hutchins' (1995) view of distributed cognition is used as a framework for understanding how individual teachers' knowledge is involved in interacting with technology to produce the mathematical knowledge made available in the classroom. Detailing how and to what extent knowledge is distributed across teacher and technology is a means of describing the participatory relationship (Remillard, 2005). This theoretical framework is depicted in Figure 1.1 in a diagram based upon Remillard's (2005) model.

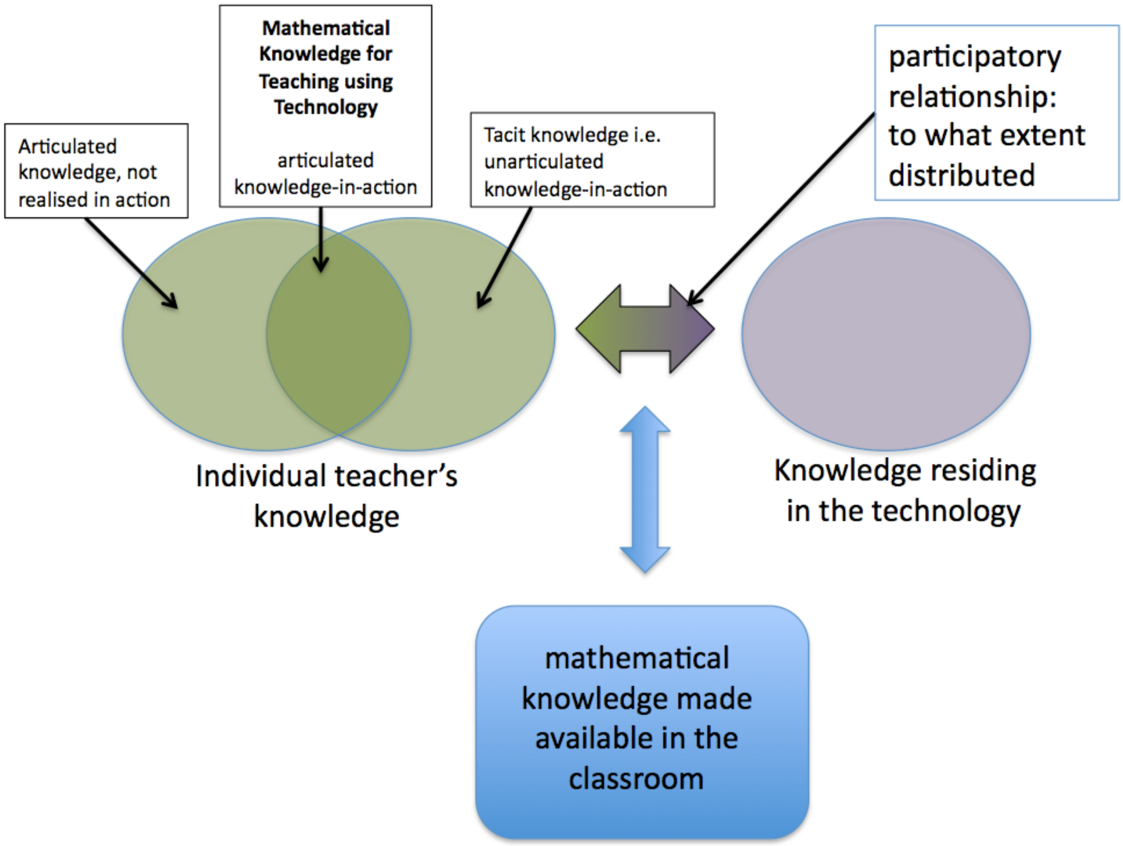


Figure 1.1 Theoretical framework for this study, based on Remillard (2005)

Hutchins (1995) presents his view of distributed cognition as a means of conceptualising cognition in a way that is "as applicable to events that involve the interaction of humans with artefacts and with other humans as it is to events that are entirely internal to individual persons" (p. 118). Critically, Hutchins argues that conceptualising cognition in this way assumes that cognition is not only a property of an individual person, but also occurs through human interaction with artefacts and other humans. In particular, he argues that cognition partially resides in tools – taken to mean any artefact appropriated for use by humans - since they incorporate in their construction the results of past cognitive efforts. The strength of his argument for a distributed view of cognition lies in his specification of minimum knowledge requirements for individual persons to carry out the computational tasks necessary for navigating a military ship *in interaction* with artefacts and other humans (e.g. reading bearings, pp. 137-140). The specification of minimum knowledge requirements provides strong evidence that it is not necessary for cognition as computation of navigational tasks to occur entirely through the actions of an individual. By specifying the minimum knowledge requirements necessary for an individual to carry out a computational task in interaction with an artefact, Hutchins (1995) provides an indication of *to what extent* cognition may be distributed across both human and artefact. For example, in his description of four ways to do distance-rate-time problems (p. 147-155), he indicates how the computation could be successfully carried out whilst the person was doing less because the tool did more.

Applying Hutchin's view of distributed cognition to this study means conceptualising the mathematical knowledge made available in the classroom as being distributed across the teacher, technology and other human participants – typically pupils, but also potentially teacher-colleagues and the author herself when in the role of participant-observer or interviewer. As noted in section 1.2.1, the focus of this study is on how individual teachers' knowledge is involved in interacting with technology to produce the mathematical knowledge made available in the classroom. Hence whilst the other human participants are acknowledged and where possible taken into account, the primary focus is on understanding how the mathematical knowledge made available in the classroom is distributed across the individual teacher and the technology. Understanding how an individual teacher's knowledge is involved in interacting with technology is then a matter of investigating *to what extent* the mathematical knowledge made available in the classroom is distributed across the individual teacher and the technology. This means investigating to what extent the mathematical knowledge made available in the classroom can be accounted for through the individual teacher's mathematical knowledge for teaching using technology. In particular, specifying the minimum knowledge requirements necessary for the teacher to produce the mathematical knowledge made available in the classroom in interaction with the technology (and other human participants) provides an indication of how and to what extent cognition may be distributed across both teacher and technology.

Hence, Hutchins' (1995) view of distributed cognition provides a means of investigating how individual teachers' knowledge is involved in the participatory relationship with technology. Critically for this study, by specifying minimum knowledge requirements, Hutchins' (1995) view of distributed cognition provides a means of indicating how and to what extent the case study teachers' mathematical knowledge for teaching was distributed across the software, the author and teaching-colleagues.

In chapters 6 and 7, the terms *readerly* and *writerly* response (Bowe, Ball & Gold, 1992, drawing on the work of Barthes) are introduced to indicate how and to what extent knowledge is distributed across teacher and technology. Departing from their original meaning, that *writerly* texts invite the reader to participate in meaning-making and are therefore in a sense superior to *readerly* texts that make no such demands, the use of these terms in this study takes a less normative view. Instead, a readerly/writerly response indicates the role of the individual teachers' knowledge in the participatory relationship with technology to produce the mathematical knowledge made available in the classroom. Similarly, contrasting tacit to articulated knowledge in section 1.2.3 is not intended to set up a hard dichotomy but rather to suggest a continuum indicating the extent to which knowledge is distributed across the teacher and technology.

1.3.2 The instrumental approach

This study uses an amalgam of Remillard's (2005) perspective and Hutchins' (1995) view of distributed cognition as a means of understanding how individual teachers' knowledge is involved in interacting with technology. However, these are not the only available frameworks for understanding how individual teachers' knowledge is involved in interacting with technology. In particular, the instrumental approach has been highly influential in research on teachers and technology in mathematics education (see Chapter 2). This section briefly describes the instrumental approach and, by contrasting it with Hutchins' view of distributed cognition, justifies the author's theoretical selection.

The instrumental approach (Guin, Trouche & Ruthven, 2005) in research on technology in mathematics education was based on Verillon and Rabardel's (1995) theory of instrumented activity systems. Their model of an instrumented activity system (IAS) consists of a triad: a human subject, an instrument and an object towards which the action using the instrument is directed, as well as the interactions between the three elements of the model (Verillon & Rabardel, 1995). The IAS model was developed as a means of understanding human interactions with artefacts to perform a task. An artefact becomes an instrument as the human subject appropriates it for use to perform a given task. The process of appropriation is termed instrumental genesis. An instrument is thus considered distinct from the artefact because it represents a psychological tool, incorporating both the artefact and utilisation schemes developed though the instrumental genesis of the subject.

The instrumental approach was first used in mathematics education in relation to pupils learning to use digital technologies for solving mathematical tasks (e.g. Guin & Trouche, 1999). Verillon and Rabardel (1995) used their model in relation to pupils imagining machines that could transform prisms made out of wood into cylinders and cones; pupils learning to use lathes to perform similar transformations of wooden objects; and pupils learning to use a robot to move an object in 3-dimensional space. Due to the similarity of the research contexts, appropriating the IAS model into the field of mathematics education for the purpose of analysing pupils learning to use digital technologies for solving mathematical tasks seems straightforward, requiring little further theoretical elaboration.

Applying the instrumental approach to teachers' use of technology in mathematics education however, has been less straightforward, apparently requiring further theoretical elaboration. For example, Haspekian (2005, 2014) found it necessary to distinguish between a teacher's personal and professional instrumental genesis. That is, appropriating an artefact for the teacher's personal mathematical work produces one instrument, whilst appropriating an artefact for the professional work of teaching mathematics produces a different instrument. Another example is Gueudet and Trouche's (2009) introduction of the documentational approach as a new framework for describing a teacher's appropriation of technology. The documentational approach, mirroring the instrumental approach, replaces artefact with *resources* and instrument with *document*. Thus teachers are modelled as appropriating resources and, through a process of documentational genesis, incorporating utilisation schemes to form a document.

The centrality of the construct of instrumental/documentational genesis, i.e. the process of appropriation, in these adaptations of the instrumental approach means that these studies focus on teachers' learning in relation to using technology to teach mathematics. Whilst teachers' learning is important, it is not the primary focus of this study. Instead, this study concentrates on teachers as workers where the focus is on the knowledge they 'have', using technology in their work of teaching mathematics, rather than on teachers as learners where the focus is on how their knowledge changes and develops. Hence Hutchins' view of distributed cognition seemed more appropriate to this study because it focuses on an analysing the interaction of humans with artefacts in the workplace. In particular, by defining minimum knowledge requirements, distributed cognition provides a means of analysing how individual knowledge is involved in human-artefact interactions in the workplace rather than the process of knowledge acquisition. Hence Hutchins' view of distributed cognition was selected as a framework for understanding how individual teachers' knowledge is involved in interacting with technology to produce the mathematical knowledge made available in the classroom. The workplace Hutchins analyses in his book *Cognition in the Wild* is that of a United States Navy ship the USS Palau. There are significant differences between this workplace and that of the mathematics classroom. In particular, the work of members of the navigation team on the USS Palau was very strictly regulated in comparison to that of teachers. These regulations were imposed on the navigation team to avoid contingencies or unexpected situations arising. Such situations could result in life-threatening consequences on the USS Palau so strict regulations were critical to their work. In contrast, unexpected situations might be more welcome in the work of teaching and represent opportunities for teachers to learn (Clark-Wilson, 2010; 2014). This represents a possible limitation of the theoretical framework adopted for this study. Nevertheless, Hutchins' view of cognitive framework was selected as a framework for understanding

Remillard's (2005) perspective identifies pupils as a key factor in influencing the participatory relationship between teachers and technology. However, Verillon and Rabardel (1995) make clear that the IAS model does not cover situations of instrumented activity involving collective activity i.e. more than one subject. Thus the IAS model would not easily apply to teacher-student interactions with technology or to interactions between teachers and their colleagues with technology. Based on the IAS model, the instrumental approach in mathematics education and subsequent adaptations suffer from similar limitations. Gueudet and Trouche (2009) suggest that human-human interactions are included under the documentational approach and that they take account of teachers working collectively. However, it remains unclear how such interactions and collective work figure in their theoretical model of a document, based as it is on the original IAS model.

In contrast, Hutchins' view of distributed cognition was developed with the explicit intention that it should be "as applicable to events that involve the interaction of humans with artefacts and with other humans as it is to events that are entirely internal to individual persons" (p. 118). For example, Hutchins analyses members of the navigation team working in coordination with each other to execute tasks successfully. Hence this framework does provide a means of taking teacher-student interactions with technology or to interactions between teachers and their colleagues with technology into consideration. Although the primary focus of this study is on understanding how the mathematical knowledge made available in the classroom is distributed across the individual teacher and the technology, Hutchins' view of distributed cognition allows the other human participants to be acknowledged and where possible taken into account.

1.4 Frameworks for analysing teachers' individual knowledge

An individual teacher's mathematical knowledge for teaching using technology, as defined in section 1.2.3, is when their know-how or knowledge-in-action coincides with their articulated knowledge. Hutchins' (1995) view of distributed cognition provides a means for understanding how individual teachers' knowledge is involved in *interacting* with technology to produce the mathematical knowledge made available in the classroom. However, it does not provide a means of analysing an individual teacher's *own* knowledge in relation to using technology to teach mathematics. That is, another framework is needed to analyse and compare teachers' espoused theories (articulated knowledge) and their theories-in-action (knowledge-in-action), in order to make inferences about individual's mathematical knowledge for teaching using technology – see Figure 1.1. This section justifies the selection of the *Technological Pedagogical and Content Knowledge* (TPACK) framework (Mishra & Koehler, 2006) and the Knowledge Quartet (Rowland et al., 2005) as frameworks suitable for this purpose.

In view of the definition of mathematical knowledge for teaching and the focus on digital technologies adopted in this study, a framework for analysing an individual teacher's *own* knowledge in relation to using technology to teach mathematics should fulfil two criteria:

1) accommodates an analysis of both knowledge-in-action and articulated knowledge

2) enables a focus on teachers' mathematical knowledge for teaching in relation to technology.

The literature review in Chapter 2 provides a more detailed analysis of potential frameworks for analysing an individual teacher's own knowledge in relation to using technology to teach mathematics. The broad finding from the literature review was that frameworks from research on teacher knowledge in general and in mathematics education in particular tended to fulfil the first criterion but did not focus on technology (Ball et al., 2008; Baumert et al., 2010; Davis & Simmt, 2006; Ma, 1999; Rowland et al., 2005; Shulman, 1986). Research on teachers and technology in mathematics education tended to fulfil the second criterion but focussed either on knowledge-in-action or know-how (Drijvers et al., 2010; Gueudet & Trouche, 2009; Ruthven et al., 2008; Ruthven, 2009; Trouche, 2005) or on aspects of articulated knowledge (Bowers & Stephens, 2011; Monaghan, 2004; Pierce & Stacey, 2010; Zbiek et al., 2007) but not on both.

The TPACK framework (Mishra & Koehler, 2006) is unusual in fulfilling both criteria – hence its selection for this study. Their framework represents Shulman's (1986) conception of pedagogic content knowledge diagrammatically as the intersection of two circles representing general pedagogic knowledge and content knowledge. Extending this representation using a Venn diagram with three overlapping circles, they incorporate technology knowledge as a third domain of teacher knowledge. The inclusion of this third domain of teacher knowledge enables a focus on teachers' mathematical knowledge for teaching in relation to technology. Since the TPACK framework is informed by Shulman's (1986) concept of PCK, it is reasonable to assume that it enables analysis of individual teachers' articulated knowledge or propositional knowledge in Shulman's terms. However Mishra and Koehler also emphasise the situated (Brown, Collins & Duguid, 1989) nature of teacher knowledge in the context of classroom practice, enabling an analysis of knowledge-in-action or know-how. In Chapter 5, Noss and Hoyles' (1996; & Kent, 2004; & Pozzi, 2002) notion of situated abstraction is used to reconcile the apparent duality of an individual teacher's mathematical knowledge for teaching using technology as at once 'abstract', articulated knowledge and yet situated in the context of teaching mathematics using technology. Similarly, Adler's (1999; 2001) concepts of the dilemma of transparency and visibility/invisibility are used to provide insight on this tension.

The TPACK framework is not without limitations, sketched briefly here and in more detail in the literature review in Chapter 2. Firstly, the components of the TPACK framework appear weakly theorised and thus establishing clear distinctions between them in empirical terms has proved problematic. Secondly, due to the TPACK framework's development outside of mathematics education, it lacks of subject-specificity – this may explain in part Ruthven's (2014, p.380) criticism that it provides a rather "coarse-grained tool" for conceptualising and analysing teacher knowledge. To mitigate this second limitation, the Knowledge Quartet (Rowland et al., 2005) was also selected to complement the TPACK framework. The Knowledge Quartet was chosen as a suitable tool for analysis primarily due to the framework's development as a means of focusing attention on teachers' mathematical knowledge, hence compensating for the TPACK framework's lack of subject specificity. In addition, the Knowledge Quartet fulfils the first criterion in accommodating an analysis of both knowledge-in-action and articulated knowledge. The fourth category of the Knowledge Quartet, *Contingency*, focuses on teachers' knowledge in relation to unexpected situations arising in the mathematics classroom. This focus on knowledge arising in contingent or unexpected situations helps to ameliorate the limitations of Hutchins' view of distributed cognition mentioned in section 1.3.2. Finally, the strong face validity of the Knowledge Quartet also made the framework generally appealing to the author.

This study uses an amalgam of Remillard's (2005) perspective and Hutchins' (1995) view of distributed cognition as a means of understanding how individual teachers' knowledge is involved in participation with technology. However, the TPACK framework (Mishra & Koehler, 2006) and Knowledge Quartet (Rowland et al., 2005) complement each other to provide a dual means of focusing on and analysing individual teachers' *own* knowledge in relation to using technology to teach mathematics. In particular, whilst the TPACK framework enables a focus on teachers' mathematical knowledge for teaching in relation to *technology*, the Knowledge Quartet enables a more fine-grained analysis of *mathematical* knowledge for teaching using technology.

1.5 An imperative for exploring a connectionist orientation in relation to technology

This section begins by defining a connectionist orientation in more detail and contrasting it with the transmission and discovery orientations that Askew et al (1997) found were associated with less effective teaching of mathematics. Sub-section 1.5.1 sets out the relevance of a connectionist orientation in relation to technology use and to the context in which this study is set to argue that there is an imperative for exploring a connectionist orientation in relation to technology. Finally, the imperative for exploring a connectionist orientation in relation to technology is used to justify the two main research questions and two subsidiary research questions that this study aims to address.

In a study of primary school teachers in the UK focusing on their teaching of numeracy, Askew et al (1997) found that more effective teachers of numeracy displayed what they described as a *connectionist* orientation in contrast to the *transmission* or *discovery* orientations of their less effective colleagues. Askew et al (1997) defined these orientations mainly in terms of beliefs about what it means to be numerate and about the relationship between teaching and learning. In section 1.2.2 it was noted that such beliefs are considered as ‘knowledge’ in the terms of this study.

As a result, connectionist teachers are defined as knowing that “being numerate means having a rich network of connections between different mathematical ideas and employing teaching approaches that establish and emphasise such connections” (Askew et al., 1997, p.3). In contrast, transmissionist teachers’ knowledge of what it means to be numerate is that it concerns the acquisition of a collection of routines or procedures. As a consequence, transmissionist teachers place more emphasis on teaching as a clear explanation of routines, reducing interactions between teachers and pupils to checks that these routines can be reproduced correctly. For discovery teachers, learning takes precedence over teaching: pupils' own strategies are the most important, irrespective of how effective or efficient they are. Discovery teachers make extensive use of practical experiences to embody mathematical ideas so that pupils discover methods for themselves. Connectionist teachers seek to balance teaching and learning so that there is a dialogue between teacher and pupils, both to exemplify the teacher's network of knowledge and skills and to reveal pupils' thinking.

In terms of knowledge of what it means to be numerate, connectionist and transmissionist teachers can therefore be seen as occupying opposite ends of a spectrum, with a connected understanding of mathematics at one end versus a isolated and fragmented at the other. In terms of pedagogy, transmissionist teachers can be characterised as being teacher-centred, whilst discovery teachers occupy an extreme student-centred position. Connectionist teachers adopt a pedagogic approach that seeks to balance the best aspects of both student and teacher-centred pedagogies.

1.5.1 Justifying the research questions of this study

The first section of this chapter argued that mathematics education research has tended to focus narrowly on digital technologies such as those termed *cognitive technological tools* (Zbiek et al., 2007) or mathematical analysis software (Pierce & Stacey, 2010). This has led to the perception of a deficit in teachers' use of digital technologies. The apparent deficit exists in two senses. Firstly, mathematics teachers appear to make only occasional use of digital technologies in their teaching. Secondly, when teachers *do* make use of digital technologies, the potential of these technologies to enhance pupils' mathematical experience in the classroom is rarely realised. Adopting a wider definition of digital technologies suggests the apparent deficit in teachers' frequency of use may be over-estimated. Teachers do integrate some types of digital technology into their classroom practice. However, these types of digital technology are commonly assumed to be teacher-centred (rather than student-centred) and to maintain or even encourage existing 'traditional' pedagogies.

Remillard's (2005) perspective is a reminder that although the constraints and affordances inherent in digital technologies may help to shape its end use in the classroom, inevitably, teachers as end-users will also work to shape the technology. Thus connectionist-oriented teachers may work to shape digital technologies commonly assumed to be teacher-centred, using them in ways that conform to their own more student-centred pedagogy. If this is the case, then the apparent deficit in teachers' use of digital technologies in terms of the extent to which their potential is realised may also be over-estimated.

A connectionist orientation towards teaching mathematics has not been investigated in relation to ICT use. This is surprising given the prominence of ICT and links made between ICT and student-centred practices (see Becker et al., 1999; Law et al., 2008). Given the consensus that effective mathematics teaching should involve features that underpin a connectionist approach to teaching mathematics (see section 1.2.2), there is an imperative for research on how technology use is associated with a connectionist orientation. This study has defined a connectionist orientation mainly in terms of knowledge, for example, knowing that understanding mathematics means having a connected knowledge of the subject. Transmissionist teachers provide a contrast with connectionist teachers in this respect. This contrast provides a means of exploring connectionist teachers' use of technology through a comparison with transmissionist teachers' use of technology. In addition, the main focus of this PhD project is an examination of individual teachers' knowledge and how it is involved in interacting with technology to produce the mathematical knowledge made available in the classroom.

The imperative for research on how technology use is associated with a connectionist orientation and this study's focus on teacher knowledge in relation to using technology to teach mathematics give rise to the two main research questions, RQ1 and RQ2, and two subsidiary questions, RQ2a and RQ2b, listed below. Contrasting transmissionist and connectionist teachers' use of technology should make visible individual teachers' *mathematical knowledge for teaching using technology* and thus allow RQ2 and its subsidiary questions to be addressed. In this sense, RQ1 provides a frame of reference for the rest of the research project through an initial exploration of how a connectionist orientation may be related to technology.

RQ1 *How is a connectionist orientation towards teaching mathematics associated with teachers' frequency of use of ICT, their orientation towards ICT and their pedagogic practices involving ICT?*

RQ2 *What is the nature and content of teachers' mathematical knowledge for teaching using technology, as represented by the central TPACK construct?*

RQ2a *How does a conceptualisation of teachers' mathematical knowledge for teaching using technology suggest ways in which such knowledge could be measured?*

RQ2b *To what extent is the mathematical knowledge made available through a teachers' interaction with technology distributed across the individual teacher and the technology?*

1.6 Structure of the thesis

Chapter 2 is a review of literature in relation to mathematical knowledge for teaching using technology.

Chapter 3 sets out the methodology and research design adopted for the study, in accordance with the theoretical framework set out in this introductory chapter.

Chapter 4 reports findings from using survey data to explore associations between ICT use and a connectionist orientation towards teaching mathematics, addressing RQ1.

Chapter 5 begins to address RQ2 by investigating the nature of mathematical knowledge for teaching using technology, drawing on interview and classroom observation data from four case study teachers.

Chapter 6 investigates the content of mathematical knowledge for teaching using technology in relation to the topic of circle theorems. In particular, the analysis reports the development and post-hoc justification of a conceptualisation of this knowledge to address RQ2 and as a first step towards addressing RQ2a.

Addressing RQ2b, Chapter 7 introduces a framework for identifying instances of distributed cognition (Hutchins, 1995), developed from an analysis of the case study teachers' mathematical knowledge for teaching circle theorems using technology presented in Chapter 6. In particular, the development of this framework led to the specification of minimum knowledge requirements necessary for a teacher to produce a *choice and use of examples* and *decisions about sequencing* (Rowland et al., 2005).

Chapter 8 synthesises the findings of the four preceding data analysis chapters and considers implications for future research, policy and practice.

Chapter 2 - Literature Review

This chapter reviews the literature for two purposes. Firstly, to identify and analyse potential frameworks for analysing individual teachers' own knowledge that fulfil the two criteria described in Chapter 1. Namely, that in view of the definition of mathematical knowledge for teaching and the focus on digital technologies adopted in this study, a framework for analysing individual teachers' *own* knowledge in relation to using technology to teach mathematics should:

1) accommodate an analysis of both knowledge-in-action and articulated knowledge

2) enable a focus on teachers' mathematical knowledge for teaching in relation to technology.

The identification and analysis of potential frameworks for analysing an individual teacher's own knowledge should therefore provide a justification for the author's choice of the TPACK framework and Knowledge Quartet for this purpose as argued in Chapter 1.

Secondly this chapter aims to justify Research Questions 2, 2a and 2b by identifying gaps in the research literature, which these questions seek to address. This chapter reviews research relevant to this study across the general and subject-specific fields of research on teachers and teacher knowledge and research on technology in education. This review therefore uses the literature selectively, providing a review for, rather than of, research (Maxwell, 2006).

2.1 Introduction

The structure of this literature review attempts to reflect the distribution of the literature relevant to this study across the general fields of research on teachers and teacher knowledge and research on technology in education. Conducting the search for literature, it became apparent that there was little overlap between these fields: within handbooks they appear to be treated as distinct research domains (see, for example Anderson, 1995). In particular, although research on teacher knowledge has begun to inform research on technology in education, there is little evidence to suggest the reverse process is occurring. A similar pattern appears in research within the (subject-specific) field of mathematics education. Research on teacher knowledge in mathematics education rarely seems to inform or be informed by research on technology in mathematics education. Thus this literature review consists of three main sections. The first section explores the general field of research on teachers and teacher knowledge and its findings with respect to technology, before focusing specifically on research on teacher knowledge in mathematics education and its findings in relation to technology. The second section explores the general field of technology in education with regard to its findings on teachers and teacher knowledge, before again focusing on mathematics education, this time with regard to research on technology and its findings in relation to teachers' knowledge. Finally, the third section provides a summary and draws conclusions, which are used to justify RQs 2, 2a and 2b. The structure of this literature review is illustrated in Figure 2.1, below:

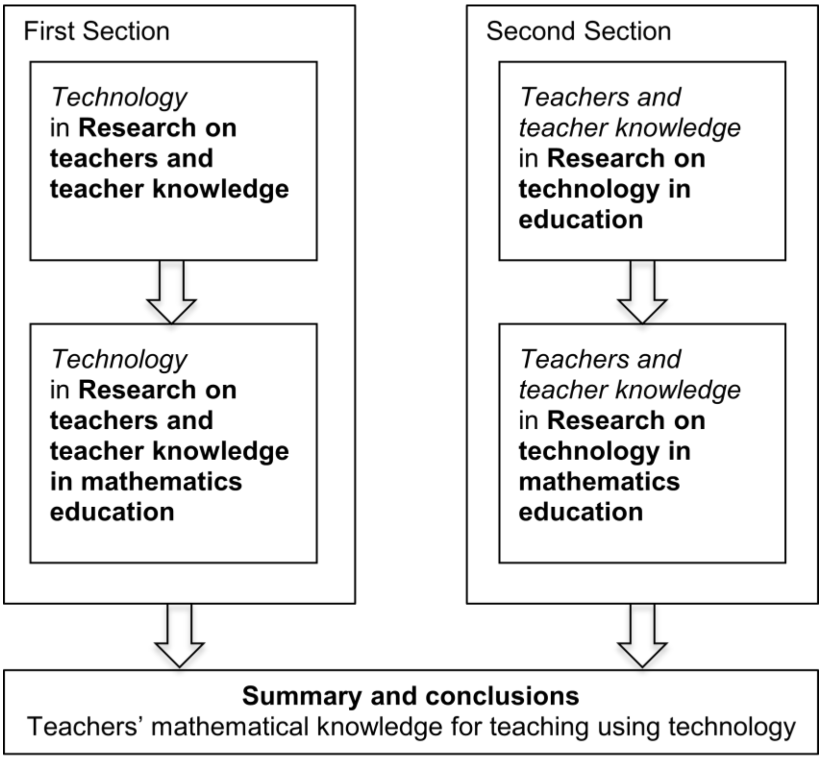


Figure 2.1 Literature review structure

2.2 Technology in research on teachers and teacher knowledge

Handbook entries in sections on teachers and teaching tend not to have technology as a focus – articles on teachers and technology, if they exist, are in sections devoted to research on technology in education (Anderson, 1995; Biddle, Good & Goodson, 1997; Guthrie, 2003; Peterson, Baker & McGaw, 2010). The *Handbook of Research on Teaching* (Richardson, 2001) exemplifies this tendency: none of its chapters has technology as a focus and its index contains only a handful of scattered references to technology or computers. Subject-specific handbooks in mathematics and science education follow this trend: reviews of research on teachers and teaching and research on technology rarely appear to coincide (Bishop, 1996; Bishop, Clements, Keitel, Kilpatrick & Leung, 2003; Clements, Bishop, Keitel, Kilpatrick & Leung, 2013; English, 2008; Fraser & Tobin, 1998; Grouws, 1992; Lerman, 2014; Lester Jr, 2007).

Technology use in teacher education does appear as a focus both in generalist and subject-specific handbooks (see for example, Musley, Lamndin and Koc (2003); Borko, Jacobs and Koellner (2010); Wiske and Spicer (2010) and Tittle (2006) on assessment of teacher learning). However, pre-service and in-service teachers' learning about how to integrate technology was not the focus of such research. Instead the focus on technology in the context of teacher education lies in how technology may be used by teacher educators to help pre- and in-service teachers engage with wider issues of (subject-specific) pedagogy, for example through internet-based communities, interactive media, video case studies and digitised work (Borko et al., 2010; Lampert & Ball, 1998; Musley et al., 2003; Wiske & Spicer, 2010). Such uses of technology in teacher education may contribute indirectly to developing teacher knowledge for teaching using technology, however this does not appear as an explicit focus in teacher education research. For example, in an introduction to a special issue of the *Journal of Teacher Education* on technology use in teacher education, Borko et al (2009) acknowledge that exploration of the knowledge and skills that teachers must have to use technology effectively in their teaching was not addressed in the issue's call for manuscripts. Attending to this lacuna, Borko et al (2009) describe TPACK (technological pedagogical content knowledge, Mishra & Koehler, 2006) as a promising construct for directing such an exploration of teacher knowledge for technology integration. The TPACK framework and its limitations are discussed in detail in the second half of this literature review, which focuses on reviewing research on teachers and teacher knowledge in the field of technology in education. Finally, Borko et al (2010) states that research on contemporary professional development programs (whether they include new technologies or not) falls mostly within the initial proof-of-concept phase of research: few studies move beyond this phase to examine the impact of professional development on teachers' learning and classroom practice.

Research on teachers and teaching does focus on teachers' knowledge, beliefs and affect, but within such research little attention is given to teachers' knowledge in relation to technology. For example, Munby et al's (2001) review of research entitled *Teachers' knowledge and how it develops*, which summarises reviews in previous editions of the *Handbook of Research on Teaching* and the *Handbook of Educational Psychology*, makes no reference to teachers' knowledge in relation to technology. In the most recent edition of the *Handbook of Educational Psychology*, Hoy et al's (2006) review of research on teacher knowledge and beliefs likewise contains no references to the knowledge teachers need for technology integration. Describing the content of teacher knowledge, Grossman (1995) and Putnam (2003) both present a list of categories based around Shulman's (1987) categorisation of the professional knowledge base for teaching. Similarly, Meijer (2010) quotes Shulman (1987, p. 8) in her portrayal of the content of teacher knowledge, listing his seven categories as follows:

- *content knowledge*;
- *general pedagogical knowledge*, with special reference to those broad principles and strategies of classroom management and organization that appear to transcend subject matter;
- *curriculum knowledge*, with particular grasp of the materials and programs that serve as "tools of the trade" for teachers;
- *pedagogical content knowledge*, that special amalgam of content and pedagogy that is uniquely the province of teachers, their own special form of professional understanding;
- *knowledge of learners* and their characteristics;
- *knowledge of educational contexts*, ranging from the workings of the group or classroom, the governance and financing of school districts, to the character of communities and cultures; and

- *knowledge of educational ends,* purposes, and values, and their philosophical and historical grounds.

Putnam and Borko (1997) choose to focus on general pedagogic knowledge, subject matter knowledge and pedagogic content knowledge as the three most researched areas of teacher knowledge, although they include curriculum knowledge as part of pedagogic content knowledge. Of Shulman's categories, curriculum knowledge appears most clearly related to teachers' knowledge of technology. Shulman (1986) describes curriculum knowledge as knowledge of the "full range of programs and associated materials designed for teaching particular subjects and topics at a given level, the variety of instructional materials available in relation to those programs" (p. 10), as well as recognising when it may be appropriate to deploy such materials. Beyond this brief description, neither Meijer (2010) nor Grossman (1995) provide further insight into what teachers' knowledge with regard to technology might entail. Putnam (2003) notes that much of what teachers know is connected to technology, referring to textbooks and instructional materials, although he does not explicitly mention digital technologies. The brevity of the description of teachers' knowledge of technology is perhaps not surprising: Putnam and Borko (1997) observe that research on pre-service and experienced teachers' knowledge and use of curriculum materials is sparse. In contrast, Van Driel and Berry (2010) state that pedagogic content knowledge has received considerable attention in the research literature, however their summary of this literature does not refer to teachers' knowledge of technology. Despite the concentration of research on pedagogic content knowledge, Van Driel and Berry (2010, p. 657) note that no universally accepted conceptualisation of pedagogic content knowledge has emerged, thus "that which is searched for and that which is uncovered is variable indeed". Research on teachers' knowledge, in particular Shulman's categorisation and its variants, arose from attempts (mainly in the US) to identify and establish a professional knowledge base for teaching, seen as integral in contributing to the professionalisation of teaching (Grossman, 1995; Meijer, 2010; Munby et al., 2001; Putnam & Borko, 1997; Shulman, 1986; 1987). As such research emphasises the critical role of professional knowledge and beliefs in teaching, it essentially makes an individualistic assumption that, for knowledge to be influential in teaching, it has to remain internal to the teacher (Putnam & Borko, 1997). Thus individualistic approaches to teacher knowledge tend to dominate, characterised by the implicit assumption that teachers 'carry' the knowledge and skill they need for teaching 'in their heads' (Putnam

& Borko, 1997). Significant moves have been made to recognise and capture the complex, contextualised nature of teachers' knowledge, through the development of a range of conceptualisations of teacher knowledge including, for example, craft knowledge (Meijer, 2010; Putnam, 2003), personal practical knowledge (Meijer, 2010; Putnam, 2003), event-structured knowledge (Putnam, 2003), case and strategic knowledge (Shulman, 1986), narrative forms of knowing (Grossman, 1995). Specifically these efforts reflect a more general movement in psychology and education towards viewing knowledge as *situated*, and teachers' knowledge in particular as situated in classroom practice (Putnam, 2003). Nevertheless, most research on teacher knowledge remains underpinned by the same essentially individualistic assumptions mentioned above (Putnam & Borko, 1997). In particular, research on teacher knowledge informed by views of cognition as *distributed* across persons and tools (Hutchins, 1995; Lave, 1988) remains underdeveloped (Putnam & Borko, 1997).

2.3 Technology in research on teachers and teacher knowledge in mathematics education

In mathematics education, research on teachers and teacher knowledge follows similar trends to those outlined above in the wider education literature. Sections on teachers and teaching in handbooks of research on mathematics education do focus on teachers' knowledge, beliefs and affect, but again within such research little attention is given to mathematics teachers' knowledge in relation to using technology in their teaching. Although there are brief reports that using technology can lead to changes in teachers' mathematical knowledge and beliefs, these tend to be based on evidence from research on technology in mathematics education, rather than research within the field of teacher knowledge, and are therefore discussed in more detail in the second half of this literature review. Thus research on teacher knowledge in mathematics education tends to neglect technology, both in terms of the knowledge needed to teach mathematics effectively using technology and as a tool for investigating mathematical knowledge in teaching. For example, in the *Handbook of Research on Teaching* (Richardson, 2001), Ball et al's (2001) article reviewing research on teaching mathematics, delineating the "unsolved problem of teachers' mathematical knowledge", does not contain any references to teachers' knowledge of technology. In the *Handbook of Research on Mathematics Teaching and Learning* (Grouws, 1992), reporting the state of the field of research on mathematics teachers' knowledge, Fennema and Franke (1992) distinguish the following components of knowledge: knowledge of mathematics; knowledge of mathematical representations; knowledge of students' learning; and general knowledge of teaching and decision-making. They mention technology only fleetingly in the context of outlining the curricular knowledge component of Shulman's model of teacher knowledge (Fennema & Franke, 1992). Similarly, articles in the same volume reviewing research on teachers' beliefs and conceptions (Thompson, 1992) and mathematics teaching practices and their effects (M. Koehler & Grouws, 1992) do not mention technology in any detail. In the *Second Handbook of Research on Mathematics Teaching and Learning* (Lester Jr, 2007), Hill et al (2007) note that new technologies such as multimedia may provide a promising means of assessing teachers' mathematical knowledge. However, they do not refer to assessments focusing on mathematics teachers' knowledge for teaching using technology. Concluding a section on teachers' beliefs about technology in his article on mathematics teachers' beliefs and affect, Philipp (2007) summarises that teachers may not believe using computers to teach is appropriate even if they are comfortable in using them for their own learning and that teachers' beliefs about technology are constrained by their beliefs about mathematics and their beliefs about teaching and learning. Sowder (2007) briefly mentions technology in relation to adopting a focus on curriculum as an approach to teacher learning, citing Bowers and Doerr (2001) as an example of a project where technology was used to develop teachers' mathematical understanding of change. Similarly, in the *Second International Handbook of Mathematics Education* (Bishop et al., 2003), Cooney and Wiegel (2003) note that technology can be used as a catalyst to restructure teachers' understanding of mathematics, quoting Laborde (2001) as an example; however, they do not explicitly address teachers' mathematical knowledge for teaching using technology. In the most recent, third edition of the *International Handbook of Mathematics Education* (Clements et al., 2013), the chapters relating to teachers' knowledge and learning (e.g. White et al., 2013) do not address teacher knowledge or learning in relation to using technology to teach mathematics. Ponte and Chapman (2008) also cite several studies (Bowers & Doerr, 2001; Goos, 2005; Ponte, Oliveira & Varandas, 2002) suggesting that technology may be used to develop pre-service mathematics teachers' knowledge but also do not address teachers' mathematical knowledge for teaching using technology directly. Boero, Dapueto and Parenti (1996) briefly state that the advent of new technologies requires new competencies on the teacher's part, to take advantage of the new educational opportunities they offer, however they do not expand on what these competencies might entail. Even and Tirosh (2008) make no mention of technology in their review of research on teachers' knowledge and understanding of students' mathematical learning and thinking. The entries in the *Encyclopedia of Mathematics Education* (Lerman, 2014) relating to teacher knowledge, the assessment of teacher knowledge and teachers as learners also do not mention teachers' knowledge or learning in relation to technology. Finally, in two recent volumes reviewing research on mathematical knowledge in teaching (Rowland & Ruthven, 2011; Sullivan & Wood, 2008), not one of the papers discusses teacher knowledge directly in relation to teaching with technology. The last section of Rowland and Ruthven's (2011) volume *Mathematical Knowledge in Teaching* focuses on building mathematical knowledge in teaching through theorised tools. Ryan and Williams' (2011) paper is the only one that employs technology in their methodology for exploring teachers' knowledge. They use *Quest* software to produce a *mathsmap* of teachers' subject knowledge and use this as a basis for individual reflection to build knowledge. However, the items the teachers engaged with to create the mathsmap did not involve technology and hence this tool is unlikely to build or give insight into teachers' mathematical knowledge for teaching using technology. Similarly, the papers in the last section of Sullivan and Wood (2008) focusing on the assessment of, and research on, teacher knowledge do not contain any references to technology.

Much of the research conceptualising teachers' mathematical knowledge in teaching is informed by or draws upon Shulman's (1986; 1987) categorisation of teacher knowledge (Ruthven, 2011). In particular, the concept of *pedagogic content knowledge* (PCK) has been the most influential of Shulman's three areas of content-related knowledge (Petrou & Goulding, 2011). Shulman (1986) describes PCK as including knowledge of an armoury of different ways of formulating and representing mathematics to make it most comprehensible to others; knowledge of what makes specific topics easy or difficult to apprehend and the conceptions (and misconceptions) that students of a certain age bring to a topic, as well as strategies for helping students overcome their misconceptions. Research effort has centred on elaborating the concept in the context of mathematics, refining and distinguishing it from other areas of content knowledge and developing measures of teachers' PCK. For example, Petrou and Goulding (2011) attempt to synthesise three theoretical frameworks for analysing teachers' mathematical knowledge in teaching: Fennema and Franke's (1992) conceptualisation described in their article (noted above) in the *Handbook of Research* *on Mathematics Teaching and Learning*; Ball, Thames and Phelps' (2008) model developed through an extensive programme of research and development at the University of Michigan; and the Knowledge Quartet (Rowland et al., 2005) developed by a research team at the University of Cambridge. Petrou and Goulding's attempted synthesis of these frameworks re-emphasises curriculum knowledge, alongside content knowledge and pedagogical content knowledge, reverting to something close to the original Shulman categorisation (Ruthven, 2011). Each of the three frameworks discussed draws to some degree on Shulman's work to inform their design. Fennema and Franke's (1992) model extends the subject-related components of Shulman's categorisation by incorporating teachers' beliefs and highlighting *knowledge of learner cognitions*. Ruthven (2011, p. 84) suggests that possibly the most important feature of this model was Fennema and Franke's (1992, p. 162) insistence on "the interactive and dynamic nature of teacher knowledge" and the need to examine it "as it occurs in the classroom".

Ball, Thames and Phelps' (2008) model, in refining Shulman's categorisation, aims to lay the foundation for a practice-based theory for mathematical knowledge for teaching. Their model retains Shulman's distinction between PCK and content knowledge, reconstituted as *subject matter knowledge* (SMK); however curricular knowledge is provisionally subsumed as a sub-category of PCK and renamed *knowledge of content and curriculum*. In addition, PCK contains two further sub-categories: *knowledge of content and students*, "an amalgam, involving a particular mathematical idea or procedure and familiarity with what students often think or do", and *knowledge of content and teaching*, also "an amalgam, involving a particular mathematical idea or procedure and familiarity with pedagogical principles for teaching that particular content" (Ball et al., 2008, p. 401-402). SMK is also split into three sub-categories: *common content knowledge* (CCK); *specialised content knowledge* (SCK) and a more tentative sub-category entitled *knowledge at the mathematical horizon*. CCK is defined "as the mathematical knowledge and skill used in settings other than teaching" – it is common in that it is used in a wide variety of settings, not unique to teaching (Ball et al., 2008, p. 399). SCK appears as a central idea in the model proposed, consisting of a unique body of decompressed mathematical knowledge and skill specialised to teaching (Ball et al., 2008). Finally, horizon knowledge is "an awareness of how mathematical topics are related over the span of mathematics included in the curriculum" and "the vision useful in seeing connections to much later mathematical ideas" (Ball et al., 2008,

p. 403). Ball et al's (2008) model was developed from a study of the way mathematical knowledge plays out in classroom practice and contributed to the development of operational measures of teacher knowledge: their test items are used throughout their paper to illustrate their view of mathematical knowledge for teaching. In particular, Hill, Rowan and Ball (2005) used a series of multiple choice items as a measure of teachers' mathematical knowledge for teaching, providing evidence to support their argument that teachers' mathematical knowledge is related to students' achievement. This was a significant result in itself since such evidence had proved surprisingly elusive and previous studies had largely failed to establish that teachers with sound mathematical knowledge are more effective than those with a limited knowledge of mathematics, at least as measured in terms of academic mathematical qualifications (Askew et al., 1997). In an influential study, Liping Ma (1999), one of Deborah Ball's students at the University of Michigan, used test items developed by Ball for her dissertation research in interviews with elementary teachers from China and the United States. Comparing the teachers' mathematical knowledge she concluded that the teachers in the US had a limited knowledge of mathematics in comparison with their Chinese counterparts. In particular, Ma (1999) articulated her conception of what it means to have a *profound understanding of fundamental mathematics*, characterising it in terms of *depth*, *breadth* and *thoroughness*. Ma (1999) defines depth as the ability to connect topics to the large, powerful ideas of the domain; breadth refers to the ability to make connections among ideas of similar or less conceptual power and thoroughness is required to weave ideas into a coherent whole. Her characterisation of mathematical knowledge in these terms serves to emphasise the importance of having a connected knowledge of mathematics for teaching. In a study of primary school teachers in the UK focusing on their teaching of numeracy, Askew et al (1997) found that more effective teachers of numeracy displayed what they described as a *connectionist* orientation in contrast to the *transmission* or *discovery* orientations of their less effective colleagues. In their model of the interplay and relationship between beliefs, knowledge and classroom practices, informed in part by Shulman's (1987) work, they define *pedagogic content knowledge* as comprising numeracy subject knowledge, knowledge of how pupils learn numeracy and knowledge of numeracy teaching approaches (Askew et al., 1997). Askew et al (1997, p. 69) found there was little to distinguish more and less effective teachers in terms of "their understanding of the content of the numeracy curriculum as far as correctness and a very straightforward sense of meaning were concerned". Thus they define their three orientations mainly in terms of the teachers' beliefs – though in the terms of this study, these are taken to be part of teacher knowledge (see Chapter 1). Specifically, connectionist teachers believe that being numerate means having a rich network of connections between different mathematical ideas and employ teaching approaches that establish and emphasise such connections (Askew et al., 1997, p. 3). However, Askew et al (1997, p. 69) did find “the connectedness of the teachers’ mathematical knowledge in terms of the depth and multi-faceted nature of their meanings” was a factor that appeared to be associated with pupil gains. None of the teachers in this study displayed a profound understanding of fundamental mathematics as described by Ma (1999). Nevertheless Ma’s (1999) findings are coherent with Askew et al (1997) in their joint emphasis of the importance of connected knowledge in mathematics.

Finally, the third framework discussed by Petrou and Goulding, the Knowledge Quartet emerged from research aimed at developing an empirically-based conceptual framework to guide lesson review discussions between teacher-mentor and student-teacher in the practicum placement of the Postgraduate Certificate in Education course in the UK (Rowland et al., 2005; Rowland et al., 2009; Turner & Rowland, 2011). The purpose of developing such a framework was to focus these discussions on the *mathematical* content of the lesson and the role of the student-teacher’s mathematics subject matter knowledge and pedagogical content knowledge. The Knowledge Quartet was initially developed from 24 lesson observations of student teachers, training to teach at primary level. These observations generated 18 codes relating to the student teachers’ classroom actions that appeared significant in the sense that they were informed by the trainee’s SMK or PCK. The codes were then grouped into four super-ordinate categories, named *foundation*, *transformation*, *connection* and *contingency*. The foundation category consists of *propositional* knowledge (Shulman, 1986) of mathematical concepts and the relationships between them and of significant research findings regarding the teaching and learning of mathematics (Rowland et al., 2005). The second category of transformation refers to knowledge-in-action, concerning the ways that teachers make what they know accessible to learners: this category focuses in particular on their choice and use of representations and examples (Rowland et al., 2005). Connection also refers to knowledge-in-action, regarding the manner in which the teacher unifies and draws out coherence in the subject matter by making connections between different concepts, representations and procedures and the decisions made by the teacher regarding appropriate sequencing of topics. Rowland et al (2005) draw parallels between the connection dimension of the Knowledge Quartet and the emphasis placed on having a connected knowledge of mathematics by Ball (1990), Askew et al (1997) and Ma (1999). Contingency concerns the teacher's ability to 'think on one's feet', to provide an appropriate response to unanticipated pupil contributions, and also notable 'in-flight' teacher insights (Thwaites, Jared & Rowland, 2011). The framework has subsequently been examined in classrooms at secondary level (Thwaites et al., 2011) and in classrooms outside the UK, specifically in Ireland and Cyprus (Turner & Rowland, 2011), resulting in the addition of new codes and alteration of some of the original codes. The Knowledge Quartet acknowledges links to Shulman's categorisation of teacher knowledge and his six-stage cycle of pedagogical reasoning (Shulman, 1987), which has been comparatively neglected in the field relative to his categorisation (Ruthven, 2011). However, in contrast to the other two models discussed by Petrou and Goulding, this model does not seek to extend or refine Shulman's work (Ruthven, 2011). Instead, the Knowledge Quartet essentially provides a heuristic tool to guide attention to and aid the analysis of mathematical knowledge-in-use in teaching (Ruthven, 2011). In particular, the Knowledge Quartet differs from the other two models discussed by Petrou and Goulding in that "the distinction between different kinds of knowledge is of lesser significance than the classification of situations in which mathematical knowledge surfaces in teaching" (Turner & Rowland, 2011, p. 196). In this sense, the Knowledge Quartet may be better suited to analysing teachers' knowledge-in-action or theories-in-action (Argyris & Schon, 1974, see Chapter 1) than Ball et al's (2008) or Fennema and Franke's (1992) frameworks. However, the Foundation category in particular also allows for an analysis of teachers' espoused theory or articulated knowledge, even if this is not realised in classroom practice.

All of these studies mentioned above effectively retain to some degree the essentially individualistic assumptions underlying Shulman's original model, since they emphasise the critical role of professional knowledge and beliefs in teaching and that, for knowledge to be influential in teaching, it has to reside with the teacher. Indeed Petrou and Goulding (2011, p. 23) acknowledge the "largely individualistic assumption which underpins" each of the models they discuss. Although Fennema and Franke (1992) and Ball et al (2008) insist upon the need to examine teachers' knowledge as it occurs in the classroom, essentially recognising the *situated* nature of mathematical knowledge in teaching, their conceptualisations do not appear to extend to considering teachers' knowledge as *distributed* (Hutchins, 1995) across persons and tools. It is worth noting here that Shulman (1986) himself conceived of knowledge involving propositional, case and strategic aspects. His notions of case and strategic knowledge actually move some way towards recognising the situatedness of teachers' knowledge in classroom situations and its dynamic nature in the face of classroom perturbations (Hodgen, 2011). The Knowledge Quartet also recognises the nature of mathematical knowledge for teaching as situated but it appears to adhere more closely to this assumption than either of the other two models reviewed by Petrou and Goulding (2011). For example, despite Ball et al's (2008) insistence that their model lays the foundation for a practice-based theory for mathematical knowledge for teaching, the abstract quality of their categories and the categories apparent separation from each other tends towards an impression of static "knowledge", for which Shulman's original categorisation has similarly been criticised (Fennema & Franke, 1992). By focusing on classroom actions, the original codes of the Knowledge Quartet remain closer to practice and therefore retain a greater sense of dynamism, of "knowing" in practice, than either Ball et al (2008) or Fennema and Franke's (1992) models. Of course, there is a risk that if the super-ordinate categories of the Knowledge Quartet are considered in detachment from their grounding in the original codes then this sense of dynamism might be lost or reduced (Ruthven, 2011). In addition, although Rowland et al (2005, p. 260) make use of an acquisition metaphor, implying individualist assumptions about knowledge by describing their foundation category as being about "knowledge possessed", Turner and Rowland (2011, footnote on p. 200) suggest that "this 'fount' of knowledge can also be envisaged and accommodated within more distributed accounts of knowledge resources". Alternatively, Davis and Simmt (2006) present a model for what they term teachers' *mathematics-for-teaching,* consisting of four intertwining and nested aspects, namely *mathematical objects, curriculum structures, classroom collectivity* and *subjective understanding*. The third aspect, classroom collectivity, emphasises how their conception of mathematics-for-teaching is considered as 'shared' knowledge, which they take to be synonymous for 'distributed' knowledge – at least across persons. Unlike Rowland et al (2005) and Ball et al (2008), whose models of teacher knowledge were developed through observations of mathematical knowledge arising in the context of actual classroom practice, Davis and Simmt (2006, see also Davis & Renert, 2009a) develop and illustrate their model with data arising from an in-service training session. Although they discuss data from classroom observation in relation to their tool of *concept study* (Davis & Renert, 2009b), they do not relate this data to the four aspects of their model. Thus it is unclear how their conception of mathematics-for-teaching might relate to mathematical knowledge *in teaching*, as it plays out in the classroom, and how such knowledge might be distributed across technology. Nevertheless, although Hodgen (2011) presents a convincing argument for conceptualising mathematical knowledge in teaching as situated, social and distributed, Ruthven (2011) asserts that the evidence base for this conceptualisation remains slender and fragmentary. Thus research on teacher knowledge in the field of mathematics education appears to follow a pattern similar to the wider field of education in which research on teacher knowledge informed by views of cognition as *distributed* (Hutchins, 1995) across persons and tools remains underdeveloped.

The spur for Petrou and Goulding's (2011) re-emphasis of curriculum knowledge in the synthesis of the three frameworks noted above was their experience of transferring the Knowledge Quartet from an English context to a Cypriot one, in particular, by their recognition that the importance of textbooks in the school mathematics context of Cyprus was not reflected in the Knowledge Quartet. Thus Petrou and Goulding (2011) emphasise (the cultural) context as an important factor in their model. They include in this 'context' the educational system, the aims of mathematics education, the curriculum and its associated materials (such as textbooks) and the assessment system within a given country, although they acknowledge that teachers' contexts are also local, including "the resources, both material and human, existing in their school or locality as well as the practices and ethos of the workplace" (Petrou & Goulding, 2011, p. 21). Ruthven (2011) notes that the contextual element of Petrou and Goulding's framework remains relatively under-developed, however other researchers also highlight the importance of cultural context in understanding teachers' mathematical knowledge in teaching. For example, Andrews (2011) presents evidence that the SMK and PCK required of teachers varies across geographic boundaries according to country curricula. He argues further that frameworks such as the Knowledge Quartet (Rowland et al., 2005) and Ball et al's (2008) model of mathematical knowledge for teaching are a consequence of the particular cultural contexts they were developed in, drawing on both systemic imperatives and didactic folklore (Andrews, 2011, p. 99). Similarly, Pepin (2011) contends that the work of teaching differs from country to country. Rowland (2014) summarises the issue of cultural context in relation to frameworks for conceptualising mathematics teacher knowledge, such as Ball et al's categorisation and the Knowledge Quartet, noting it is important to bear in mind that they originate from Anglo-American culture. He gives examples of other cultural emphases and influences, in parts of Europe and the Far East, which "significantly shape ways of thinking about how teachers develop as professionals" although they do not directly address mathematics teacher knowledge. In addition to cultural context, Andrews (2011) suggests the context of the mathematical topic under scrutiny may influence the apparent depth of mathematical knowledge for teaching displayed by the teacher. Much of the research conceptualising mathematics teacher knowledge has been limited in terms of its curriculum coverage, for example, Ma (1999) focused on arithmetic operations and Askew et al (1997) focused on numeracy. Finally, Williams (2011) provides evidence that teachers' knowledge may be highly sensitive to the methodological tools through which it is measured: teachers in his study were more able to articulate their knowledge in the context of a task than in researcher interviews.

2.3.1 Measurement of teacher knowledge in mathematics education

Relatively sophisticated measures of teachers' mathematical knowledge for teaching have been developed in research on teacher knowledge within the field of mathematics education (Baumert et al., 2010; Blomeke & Delaney, 2012; Hill et al., 2005; Izsak et al., 2012; Mesa & Leckrone, 2014; Tatto et al., 2012). The impact that the contextual factors highlighted in the previous section, such as cultural context, mathematical topic and methodological tools, have on teacher knowledge has important implications for the development of measures of mathematical knowledge for teaching. For example, in a study of secondary school teachers' mathematical knowledge in teaching in Germany, Baumert et al (2010) developed separate written tests to measure teachers' content and pedagogic content knowledge. Baumert et al (2010) succeeded in theoretically and empirically distinguishing between content knowledge and pedagogic content knowledge, as measured by their test items, also providing evidence that both are important factors in the quality of teaching and student learning. In particular, Baumert et al (2010) found that the positive effect for PCK on students' learning was mediated by the quality of instruction, in terms of providing cognitively challenging tasks and learner support. This mediation effect was not observed for content knowledge. Their test items for content knowledge covered arithmetic (including measurement; 4 items), algebra (2 items), geometry (1 item), functions (1 item), probability (1 item) and geometry, functions and algebra (4 items). The test of pedagogic content knowledge comprised items regarding the teachers' ability to identify multiple solution strategies to *tasks*, to recognise the difficulties, misconceptions and solution strategies of their *students* and their knowledge of *instruction* in terms of different representations and explanations of standard mathematics problems. Their test items were all open-ended, no multiple choice items were used and the tests conducted by trained administrators in single-interview situations. In contrast, using the Michigan test items, Hill et al (2008) were unable establish a clear distinction between KCS and subject matter knowledge, although as noted previously, Hill et al (2005) managed to provide the first strong evidence linking teachers' mathematical knowledge to student achievement. Again in contrast with Baumert et al's (2010) study, Hill et al's (2005; 2008) test items were multiple choice and the tests were administered through questionnaires rather than in an interview format. Their test items were aimed at primary school teachers and were limited to the three mathematical content areas they judged as most frequently taught: number concepts, operations, and patterns, functions, and algebra (Hill et al., 2005). Thus these two measures of teacher knowledge differ in terms of the cultural contexts in which they were developed, the methods with which they were applied and the mathematical topics that they cover. Research on constructing measures of teachers' mathematical knowledge in teaching has only recently focussed on exploring the impact of cultural context (see for example, Delaney et al., 2008; Blomeke & Delaney, 2012 in the special issue in the 44th volume of *ZDM – The International Journal on Mathematics Education* (Ball et al., 2012) on assessment of teacher knowledge). Hence this area still requires substantial development and further research is needed focussing on the effects of cultural context, mathematical topic and the methodological tools on the measurement of mathematical knowledge for teaching. In particular, the development of test items embedded in a technological context to measure teachers' mathematical knowledge for teaching appears to have been neglected (Mesa & Leckrone, 2014).

2.4 Teachers and teacher knowledge in research on technology in education

In handbook entries on technology in education teachers often appear in the context of limited use of technology in classroom practice. Frequently based on evidence from teacher surveys, teacher knowledge emerges as a factor explaining the success or failure of technology integration, rather than as an explicit focus of research. The knowledge teachers need to teach using technology has seldom been explored in depth. The range of terms used to refer to teacher knowledge exemplifies its inadequate conceptualisation in research on technology in education, as well as shifting trends in teacher education. Where research on technology in education mentions teacher knowledge explicitly, it is often as one amongst a range of other barriers or obstacles to successful technology integration operating at the individual, institutional and system levels. For example, Salomon (1992) follows Cuban's (1986) cycle of technology innovation by detailing the hopes for technology, the dramatic spread of access to hardware and then the reality of underuse. Drawing on Becker's (1985) study, Salomon (1992) notes that lack of teacher training in the use of computers may be a key explanatory factor in the under-use of computers, supported by teachers' self-reported lack of knowledge. Unwin and MacAleese (1988) and Becker (1992) both criticise teacher education for failing to keep up with the fast pace of development in computer technology, leaving teachers feeling under-prepared for teaching with computers compared to other aspects of practice. Similarly, having discussed improvements in access and the opportunities for 'problem-based learning' created by digital technologies, S. M. Williams (2003) notes that only one third of teachers feel prepared to use technology effectively. Lack of teacher knowledge and inadequate training is first on Hativa's (1995) list of obstacles to the proper integration of computers into classroom teaching. Noting that most teachers, teacher educators and policymakers were raised in an analogue world, Sancho (2010) identifies updating educators' knowledge of digital technologies as one of the key challenges to technology integration, amongst other systemic factors constraining the use of ICT. Hadley and Sheingold (1993) also report teachers' doubts, lack of interest or knowledge about computers as one of the main barriers to integration reported by teachers, although this factor decreased over time as teachers increased in expertise. Surveying mathematics teachers in the US, Manoucherhri (1999) found that non-users of computers lacked knowledge about how to use computers to improve learning, again amongst other factors. Lawless and Pellegrino (2007) criticised the reliance on self-report measures to indicate knowledge growth through professional development on integrating technology into teaching. In particular, they criticise the operationalisation of teacher knowledge, stating that self-report measures reflect teachers' confidence or self-efficacy rather than actual knowledge. However, since self-report measures were often the means of identifying teacher knowledge as a factor in technology integration in the first instance, this reliance suggests a more deep-seated weakness in the conceptualisation and understanding of what constitutes teacher knowledge in relation to technology integration.

Teacher knowledge is also mentioned implicitly, by referring to the skills (Forgasz, 2006; Law, 2010; Means & Roschelle, 2010; Mumtaz, 2000; S. M. Williams, 2003; Zammit, 1992), abilities (Collis, 1996a; Law, 2010; Zhao, Zhang & Lai, 2010), capabilities (Collis, 1996b) or competences (Law, 2010) teachers need for successful technology integration. The relationship between these terms and knowledge is not always clear: for example, at times knowledge may seem to underpin, lie alongside or encompass aspects of skill or ability; at times these terms seem interchangeable. Nevertheless, despite the variation in terms, research in technology education seems fairly consistent in describing teacher knowledge broadly in relation to three main areas. Firstly, teachers need to know how to use digital technologies. This is described variously as having ICT skills (Means & Roschelle, 2010); technical proficiency or the knowledge needed to operate hardware and software (Hativa, 1995); 'basic skills', such as being able to use word-processing, spreadsheet, presentation and Internet-browsing software, and familiarity with ICT tools and materials in the subjects they teach (Williams, 2003); the technical ability to use the necessary hardware and software (Zhao et al., 2010); and technical competence (Law, 2010). Secondly, teachers should have knowledge of how to choose appropriate software (Hativa, 1995); the ability to evaluate the reliability and usefulness of digital resources (Zhao et al., 2010); and the ability to make appropriate selection of ICT tools (Law, 2010). Finally, teachers should know how to incorporate digital resources into classroom activities (Williams, 2003); integrate hardware and software into classroom instruction (Hativa, 1995); and make appropriate use of ICT tools in different curriculum contexts for different pedagogical purposes (Law, 2010). These three areas appear roughly to mirror the first three recommendations made by the UK Department of Education and Science (1992) for the inclusion of information technology (IT) in initial teacher training, which at the time represented emerging thinking in the field of teacher education (Collis, 1996b). The recommendations state that all student teachers should develop an IT capability encompassing:

i) ability to make confident personal use of a range (albeit limited) of software packages and IT devices appropriate to their chosen subject and age range;
ii) the ability to review critically the relevance of software packages and IT devices;
iii) the ability to make constructive use of IT in their teaching;

iv) the ability to evaluate the way in which the use of IT changes the nature of teaching and learning.

Research in technology education also raises a number of other relevant teacher characteristics, related success or failure in technology integration, including beliefs about the nature of teaching and learning (Becker, Ravitz & Wong, 1999; Means & Roschelle, 2010), attitudes (Collis, 1996a; Hativa, 1995; Mumtaz, 2000), motivation (Hativa, 1995; Sancho, 2010), background experience (Collis, 1996a), confidence (Zammit, 1992), participation in professional development (Means & Roschelle, 2010) and a broader range of professional activities (Means & Roschelle, 2010). However, beyond the indications described above, teacher knowledge is rarely explored in further depth. For example, a detailed consideration of the nature and content of teacher knowledge needed for technology integration and how this knowledge interweaves with other types of teacher knowledge, such as knowledge of subject content or pedagogical considerations is rarely evident. As noted above, in research on technology in education, teacher knowledge tends to emerge as a factor explaining the success or failure of technology integration, rather than as an explicit focus of research. This may explain why links between research on teacher knowledge and research on technology in education are rare. As Kirschner et al (2008) state "the main stream teacher education research does not pay much attention to ICT while researchers studying ICT pay little attention to research conducted on teacher education". However, the brief description of teacher knowledge in relation to technology given above resonates with aspects of Shulman's (1986) *curriculum knowledge*, defined as knowledge of the "full range of programs and associated materials designed for teaching particular subjects and topics at a given level, the variety of instructional materials available in relation to those programs" (p. 10), as well as recognising when it may be appropriate to deploy such materials. In both cases 'technology' or the tools of teaching tend to be viewed as things that teachers should have knowledge *about*. For example, teachers should know of their existence and know how to use them, when and where to employ them to greatest advantage. Defined in this way, knowledge about technology or curriculum knowledge becomes another set of knowledge that needs to be "carried around inside the teacher's head", as Putnam and Borko (1997) put it. Thus in terms of teacher knowledge, research on technology in education rarely seems informed by notions of distributed cognition (Hutchins, 1995), requiring a shift in focus from the knowledge of the individual teacher to the knowledge of the teacher in conjunction with various available technologies (Putnam & Borko, 1997).

2.4.1 A new direction: teachers' knowledge as an explicit research focus

Law's (2010) article entitled *Teacher Skills and Knowledge for Technology Integration* indicates a new direction in research on technology in education with teacher knowledge as an explicit focus. Law (2010) quotes the SITES 2006 study findings that teachers' self-perceived technical and pedagogical ICT competence to be positive, significant predictors for teachers' ICT adoption in their own practice. In particular, she states that pedagogical ICT competence, described as the teachers' ability to make appropriate selection and use of ICT tools in different curriculum contexts for different pedagogical purposes, is the most crucial determinant of actual ICT use in instruction (Law, 2010). Law (2010) states that this kind of knowledge and the associated skills required of teachers is conceptualised by Mishra and Koehler (2006) within the Technological Pedagogical Content Knowledge (TPACK) framework – see also Chapter 1. Their framework represents Shulman's (1986) conception of pedagogic content knowledge diagrammatically as the intersection of two circles representing general pedagogic knowledge and content knowledge. Extending this representation using a Venn diagram with three overlapping circles, they incorporate technology knowledge as a third domain of teacher knowledge, to indicate the skills or knowledge needed to successfully operate technology, referred to elsewhere as technical competence (Law, 2010). The inclusion of technology knowledge introduces two new dyads, technological pedagogical knowledge (TPK) and technological content knowledge (TCK), representing the intersection of technology knowledge with pedagogic knowledge and content knowledge respectively, and a triad representing the intersection of all three types of knowledge: technological pedagogical content knowledge (TPACK, see Figure 2).

Koehler and Mishra (2009) define TCK as knowledge about the manner in which technology and content influence and constrain one another. They argue that teachers need to develop an understanding of how the subject matter (or the kinds of representations that can be constructed) can be changed by the application of particular technologies (Koehler & Mishra, 2009). For example in mathematics, dynamic geometry software, such as Cabri Geometry (Texas Instruments, 2007), introduces an explicit order to the construction of geometric figures that does not exist in a paper and pencil environment (Jones, 2000). Using dynamic geometry software, dragging becomes a method for verifying a geometric construction, since the geometric relationships in a correctly constructed figure will be invariant under drag. This in turn raises questions about the role of proof in geometry. Thus TCK can be conceptualised as knowledge of how software models mathematical concepts and relations and of how the software design may therefore affect both the substantive and syntactic structures of mathematics.

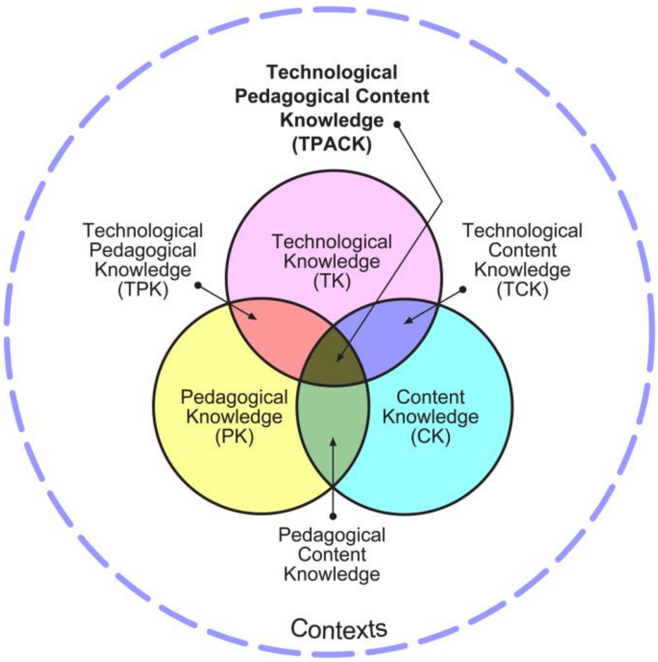


Figure 2.2 The TPACK framework, source http://tpack.org/

TPK comprises knowledge of the existence, components and capabilities of various technologies for use in teaching and learning settings and pedagogical considerations for their selection (Mishra & Koehler, 2006). For example, teachers need to be able to reinterpret the function of generic software and hardware, such as word-processing, spreadsheet or presentational software or interactive whiteboard hardware, to suit their own pedagogical purposes (Koehler & Mishra, 2009). This might include how to manage changes in the *working environment* and *activity format* (Ruthven, 2009), requiring the adaptation of strategies for classroom management and organisation. Mishra and Koehler (2006) suggest TPK would also include knowledge of technologies for maintaining class records, attendance, and grading.

Mishra and Koehler (2006) argue that TPACK goes beyond knowledge of the individual components (content, pedagogy and technology). Rather TPACK represents knowledge of the dynamic, transactional relationship between the three knowledge components. In a similar vein to Shulman's (1986) argument for pedagogic content knowledge, Mishra and Koehler (2006) suggest that TPACK is a special form of knowledge, different from that of the technology expert, subject matter specialist or the general pedagogic knowledge shared by teachers across disciplines. In teaching mathematics, TPACK could be exemplified by the knowledge underlying a teacher's selection of spreadsheet software for the capability to manipulate variables and formulae dynamically for the pedagogic purpose of supporting an investigative approach to learning algebra, whilst understanding the limitations of the mathematical representation, such as the discrepancies between spreadsheet and standard algebraic notation (Dettori et al., 1995), and recognising and developing strategies to deal with the pedagogical implications of these limitations.

Mishra and Koehler's TPACK framework is representative of a new direction in research on technology in education. Their explicit focus on teacher knowledge led them to incorporate and extend research from the field of teacher knowledge, in particular Shulman's concept of pedagogic content knowledge, to frame their conception of the knowledge teachers need for technology integration. In addition, they use situated theories of cognition, referring to (Brown, Collins, & Duguid, 1989), to inform their conceptualisation of TPACK, moving away from a strictly individualistic approach to recognise that knowledge about technology cannot be treated as context-free (Mishra & Koehler, 2006). They are not alone in focussing explicitly on teacher knowledge for technology integration (see also Cox & Webb (2004), Leach & Moon (2000), Loveless & Ellis (2001), McCormick & Scrimshaw (2001), for example), nor do they claim to be unique in arguing that knowledge about technology cannot be treated as context-free and is interwoven with knowledge of pedagogy and content (Mishra & Koehler (2006), see for example Zhao, 2003). However, their articulation of a framework specifying the relationships between knowledge of pedagogy, content and technology in terms of the dyads TCK and TPK and the triad of TPACK does distinguish their approach from that of other researchers. Hence for the present study, the TPACK framework provides a particular focus on knowledge with respect to technology that is lacking in frameworks developed in research on teachers and teacher knowledge. In addition, it seems reasonable to assume that the TPACK framework could be used to analyse both articulated knowledge and knowledge-in-action. For example, the TPACK framework is based on Shulman's (1986) concept of PCK, which was underpinned by his notions of propositional, case and strategic knowledge introduced in the same paper – although Mishra and Koehler (2006) do not refer to these notions in their description of the TPACK framework. Propositional knowledge can be roughly equated with articulated knowledge, whereas case and strategic knowledge appear to refer to knowledge-in-action.

Law (2008) suggests the TPACK framework described by Mishra and Koehler (2006) bridges the gap between the fields of educational technology and teacher education identified by Kirschner et al (2008). Similarly, Borko et al (2009) recognise Mishra and Koehler's contribution to understanding the "wicked problem" of teaching and learning with new technologies, describing the TPACK framework as a compelling conceptualisation of the unique knowledge teachers need to develop to embed technology knowledge in their instructional practice so that it fosters student learning. However Law (2008; 2010) argues it is not adequate to describe the kinds of teacher learning required for "ICT integration as a lever for innovation and transformation". In addition, to support educational transformation, Law (2008; 2010, p. 215) argues the need for teachers to develop a range of metacognitive, sociometacognitive and socioemotional abilities and capacities, requiring both courage and motivation as well as epistemological beliefs aligned with socially grounded, constructivist theories of learning. Nevertheless, the TPACK framework covers the kinds of knowledge and skills denoted by pedagogical ICT competence, which Law (2010) identifies as the crucial determinant in successful technology integration. Hence developing TPACK may be seen as a pre-requisite or at least an integral part of the process of developing the kinds of abilities and capabilities that Law (2010) suggests are necessary for ICT to act as a lever for educational transformation.

2.4.2 Problems concerning the TPACK framework

Considerable work needs to be done to contextualise constructs issuing from a generalist framework such as TPACK within specific subject domains. For example, extensive research effort has been directed towards contextualising Shulman's concept of pedagogical content knowledge, the construct underlying the TPACK framework, within mathematics education. In particular, Ball and colleagues have elaborated PCK within mathematics education (Ball, et al., 2008) and developed measures of mathematics teachers' *specialised content knowledge* which have been successfully linked to student achievement (Hill, Ball, & Schilling, 2008; Hill, et al., 2005). In contrast, few researchers have attempted, even within the field of educational technology, to create survey instruments to measure TPACK constructs (Graham, 2011) and even these have tended to rely on the type of teacher self-reports of knowledge criticised by Lawless and Pellegrino (2007) - see for example, Archambault and Barnett (2010). In addition, the TPACK framework is only beginning to be explored within research on specific subject domains (Voogt et al, 2012) and in mathematics education specifically (e.g. Bellman et al., 2014; Bowers & Stephens, 2011; Drijvers et al., 2013; Drijvers et al., 2014). A paper by Bowers and Stephens (2011) provides an example of an attempt to contextualise the framework within mathematics. They argue that PCK and TPK have previously received much research attention, whereas TCK has not. Thus they limit their examination of research to findings relating to TCK and TPACK within mathematics education. Bowers and Stephens (2011) offer three constructs that they consider as lying at the intersection of mathematical knowledge and technological knowledge but conclude that TPACK, as the intersection of knowledge of mathematics, pedagogy and technology, may represent the empty set. Instead, they suggest TPACK should be regarded as an orientation rather than a subset of particular knowledge or skills. By orientation, they mean that teachers should develop a disposition or proclivity towards viewing technology as a critical tool for identifying mathematical relationships. Citing Zbiek and Hollebrands (2008), they suggest that to develop TPACK, teachers need to become "aware of how to design rich tasks that integrate technology into the classroom discourse so that technology-based conjectures and arguments become normative" (Bowers and Stephens, 2011), thereby essentially equating a TPACK orientation with successful technology integration. In contrast, Niess et al (2009) propose TPACK as integrated knowledge, representing the intersection and interconnection of content, pedagogy and technology knowledge. The differences between Niess et al (2009) and Bowers and Stephens' (2011) conception of the central TPACK construct is illustrative of some of the theoretical difficulties besetting the TPACK framework elaborated by Graham (2011). Graham (2011) identifies a lack of coherence in the interpretation of the central TPACK construct, with some researchers taking an *integrative* perspective, viewing TPACK as the use of the distinct domains of pedagogical, content and technological knowledge in combination, whilst others take a *transformative* perspective, viewing TPACK as a new domain of synthesised knowledge that cannot be explained by the sum of its parts. Graham (2011) argues this confusion is exhibited in Mishra and Koehler's paper, where the representation of the TPACK using a Venn diagram suggests an integrative model, whilst the language they use to describe the central TPACK construct suggests instead a transformative model. Further, Graham (2011) argues that some researchers elide this tension altogether by making no distinction between TPACK and technology integration, ignoring the other elements in the model. To some degree, Bowers and Stephens (2011) fall into this camp, although they acknowledge the other model constructs, they disavow the existence of the central TPACK construct as a form of knowledge, instead essentially equating it with technology integration. Voogt et al (2012) and Ruthven (2014) echo Graham's criticisms of the weak theorisation of the TPACK framework, drawing attention to its ambiguities and lack of clarity in defining the framework's constructs and the inter-relations between them. More specifically, the TCK, PCK, TPK and TPACK constructs seem particularly ambiguous in terms of their theoretical definition (Ruthven, 2014). In addition, Ruthven (2014) suggests the TPACK framework provides "a rather coarse-grained tool" for analysing teacher knowledge and, hence, may need supplementing by other frameworks to achieve a sufficient depth of analysis.

Mishra and Koehler criticise Shulman's conception of PCK for ignoring the knowledge needed to use technology, stating he "did not discuss technology and its relationship to pedagogy and content". They argue that Shulman's PCK was limited by the time of its conception: that in the 1980s, for example, traditional pedagogical technologies achieved a *transparency* of perception such that they had become commonplace, not even regarded as technology. They suggest that the rise of new digital technologies, not yet part of the mainstream, has forced the recognition that teachers need a body of knowledge incorporating the skills and techniques needed to make use of technology (in the wider sense). Yet Shulman's (1986) *curriculum knowledge* is a recognition of a body of teacher knowledge incorporating the skills and techniques needed to make use of technology. His exemplification of a biology teacher's curriculum knowledge to include "alternative texts, software, programs, visual materials, single-concept films" etc (p. 10) suggests that digital technologies were already around to challenge the apparent status quo of traditional pedagogical technologies. The process of *transformation* indicated in Shulman's (1987) model of pedagogical reasoning focuses on the critical *interpretation* of texts (which could be taken to include technology), the *representation* of disciplinary concepts and their *adaptation* and *tailoring* to a school situation and group of students. As suggested by Ruthven (2011, p. 86), viewing transformation as a problem solving process, we see that it is subject to a range of constraints at the same time mathematical, pedagogical and curricular, which often cannot be considered in isolation. In this sense, if curricular constraints are taken as those imposed by the 'tools of the teaching trade', i.e. to include technology as in Shulman's conception of curriculum knowledge, then the process of transformation appears closer to the transformative model of TPACK than the integrative model. However, it is important to acknowledge that whilst Shulman's use of transformation refers to a *process*, the central TPACK construct is viewed as a new *domain* of knowledge from a transformative perspective. Shulman's process of transformation is central to the second category of the Knowledge Quartet, to which it gives its name. However, the Transformation category of the Knowledge Quartet groups classroom situations, where teachers' SMK or PCK was evident in making what they know accessible to learners.

Viewing transformation as a problem-solving process as described above, Ruthven (2011, p. 86) argues helps to explain "why it has been so difficult to make demonstrable progress in establishing persuasive and productive knowledge taxonomies" for mathematical knowledge in teaching. Ruthven (2011, p. 85) argues further:

> …many teaching problems cannot be adequately framed in 'pure' terms drawn from a single knowledge domain, or even by drawing on several domains independently. Put simply, satisfactory resolution of teaching problems must take account of, and often trade off between, interacting considerations of quite different types… This gives rise to solutions that often involve an irreducible fusion of such considerations, not reducible to the practice, or even logic, of any single pure knowledge domain.

In particular, applying Ruthven's argument to the TPACK framework helps to explain some of the other theoretical difficulties identified by Graham (2011). Namely, why achieving clear definitions of the individual constructs in the TPACK framework and delineating precise boundary conditions between adjacent constructs has proved problematic: Graham (2011) quotes Cox's (2008) comprehensive conceptual analysis of the TPACK research literature which found 13 distinct definitions for TCK, 10 definitions for TPK, and 89 different definitions for TPACK in the reviewed literature. Taking the transformative perspective implies viewing the central TPACK construct as an irreducible synthesis of pedagogic, content and technology knowledge that cannot be broken down into its single or paired knowledge domains. Thus the individual constructs may not actually exist in the reality of classroom practice and serve purely as a heuristic for reducing complexity to a manageable level by effacing one or more of the knowledge domains. As Mishra and Koehler (2006, p. 1029) themselves suggest, “separating the three components (content, pedagogy, and technology) in our model is an analytic act and one that is difficult to tease out in practice”. It is therefore unsurprising that Graham (2011) finds pedagogical knowledge considerations are often mentioned in the context of TCK despite the fact that PK does not contribute at all to TCK according to the framework. For example, although Bowers and Stephens (2011) identify three constructs from mathematics education relating to TCK, pedagogical considerations still emerge in their descriptions. In describing *knowledge constructed from noticing*, Bowers and Stephens (2011) mention various pedagogic techniques for encouraging noticing, including the use of colour, motion and markings during software use and the importance of experimentation, surprise and the inclusion of multiple solution pathways as features in task design. They also suggest engaging pupils in open-ended explorations or “what-if” explorations with software as profitable pedagogic strategies. Indeed their description of a TPACK orientation as one that “views technology as a critical tool for identifying mathematical relationships” implies a judgement that students may learn better with technology than they do in other environments. This judgement can only be based on pedagogical considerations involving knowledge of how students learn. This suggests that the central TPACK construct is likely to involve more than a positive stance towards technology. In addition, the transformative model of TPACK may be closer to the reality of classroom practice than an integrative model.

Finally, Mishra and Koehler’s criticism of the separation of PCK and curriculum knowledge could instead be seen as a call to investigate the *distributed* (Hutchins, 1995) nature of mathematical knowledge in teaching. Their recognition that the interaction between PCK and technological knowledge has been neglected, suggests shifting away from individualistic approaches to focus on “the interplay between teachers’ knowledge and other available resources” (Putnam & Borko, 1997), for example digital technologies. This entails thinking in terms of distributed cognition (Hutchins, 1995). In this case, rather than viewing knowledge solely as a property of an individual, it is distributed or “stretched over” (Lave, 1988) both the individual teacher and the various technologies available (Putnam & Borko, 2000). Although Mishra and Koehler invoke the situated nature of knowledge in describing TPACK, they do so in the sense that the classroom situation provides contextual shaping for a teacher’s knowledge, nevertheless knowledge remains internal to the individual teacher. Mishra and Koehler's critique of Shulman's taxonomy lies in their criticism of his distinction and hence separation of PCK and curriculum knowledge. As argued above, the transformative model of TPACK appears to draw closer to the process of transformation in Shulman's model of pedagogic reasoning and, as a result of their individualistic approach, Mishra and Koehler do not move far beyond Shulman's original taxonomy. Adopting a distributive approach, it is argued, would be more likely to provide the kind of research understandings that could be used to support teachers in their pedagogical reasoning through the use of digital technologies (Putnam & Borko, 1997).

2.5 Teachers in research on technology in mathematics education

A review of research on technology in mathematics education covering the period from 1994-98 found that very few papers considered the *teacher dimension*, that is the role of the teacher in technology integration, focussing instead on understanding the interaction between students and technology (Lagrange et al., 2003). This is reflected in the relative absence of the teacher from handbook articles on technology and mathematics education covering a similar period. In the *Handbook of Research on Mathematics Teaching and Learning* (Grouws, 1992), Kaput (1992) briefly notes the importance of support for teachers who must come to terms with the profound changes in their beliefs about mathematics, teaching, learning and students, that may be provoked by technology innovation. Citing Lampert (1988), he suggests that such changes may be akin to a "religious transformation, with all the accompanying travail and exhilaration" (Kaput, 1992, p. 548). Kaput (1992) also notes that the level of effort and expertise required to retrofit general tools and applications, such as calculators, symbol manipulators, graphing utilities and spreadsheets, as learning tools is closely related to the difficulty of technology integration and hence the limited penetration of technology into mathematics classrooms. Ruthven (1996) lists teacher confidence amongst other factors relating to the modest impact of calculators on mathematics education, but otherwise makes little reference to teachers. In the same volume, Balacheff and Kaput (1996) suggest that teachers' traditional professional knowledge is insufficient to deal with the deep changes in learning, teaching and epistemological phenomena emerging due to computer-based learning environments, but no give indication of what new knowledge it is that teachers might require. None of the papers in the section entitled

Influences of Advanced Technologies in either the first or second edition of the *Handbook of International Research in Mathematics Education* (English, 2002; 2008) has teachers' use of technology as a major focus. However, Yerushalmy and Chazan (2002) do include a section on teacher knowledge of school algebra, discussing how one teacher's efforts to integrate graphing technology into a standard approach to school algebra led to her recognition of the tension between viewing letters as unknowns in an equation or as variables in a function – this is discussed further below.

Following recognition of the complexity of integrating technology, Lagrange et al (2003) point to an emerging trend in research focusing on the role of the teacher in technology integration, through observations of 'ordinary' teachers as they struggle to integrate ICT into their classroom practice. Similarly, Hoyles and Noss (2003) identify a common research trajectory for the study of digital technologies in mathematical learning: starting with documenting potentials and obstacles in software use and then gradually shifting to discussions of tool mediation, tasks and activities and the role of the teacher. The *Second Handbook of Research on Mathematics Teaching and Learning* (Lester Jr, 2007) gives an indication of the shift in research focus: Zbiek et al (2007) identify three research constructs relating to teachers' practice involving technology, amongst 17 in total variously relating to technology and technology in connection with students and with the curriculum. Similarly, the recent 17th ICMI Study *Mathematics Education and Technology – Rethinking the Terrain* (Hoyles & Lagrange, 2010) demonstrates this shift with a section entitled *Teachers and Technology*, organised to consider a range of issues related to preparing teachers to teach using technology and to the challenges of technology integration. The publication of *The Mathematics Teacher in the Digital Era* (Clark-Wilson, Robutti & Sinclair, 2014) makes the role of the teacher in integrating digital technologies the central focus of research, bringing to the fore an explicit focus on teachers' knowledge and a range of frameworks for analysing that knowledge (e.g. see Ruthven, 2014). The special issue in the 45th volume of *ZDM – The International Journal on Mathematics Education* on *Re-sourcing Teacher Work and Interaction* (Pepin, Gueudet & Trouche, 2013) similarly makes interactions between teacher and technology the central research focus, providing frameworks to analyse these interactions at the level of the individual teacher and to consider the collective work of groups of teachers. The next section of this literature review discusses the findings of this research trend in relation to teacher knowledge.

2.6 Teacher knowledge in research on technology in mathematics education

In their synthesis of research on incorporating mathematics technology into classroom practice, Zbiek and Hollebrands (2008) distinguish teachers' conceptions or beliefs about mathematics and technology from teachers' understanding or knowledge of mathematics. In terms of the present study, beliefs and conceptions are considered as part of teacher knowledge.

2.6.1 Teachers' conceptions

Several studies detail the changes to teachers' beliefs that may be provoked by professional development either grounded in technology use or actively promoting technology integration. Reporting on a study supporting seven teachers to implement a guided-inquiry approach to geometry using the *Geometric Supposer* software, Lampert (1988) records that teachers radically altered their views of what geometry is, of what the aims of teaching it are, and the way it should be taught. In particular, this involved re-sequencing the subject matter, finding a new way to map the geometrical terrain (Lampert, 1988). Similarly, Moriera and Noss (1995) focus on describing Portuguese teachers' attitudes and views as they unfolded during the course aimed at introducing a "Logo-mathematical culture". In the *Microworlds Project,* Noss and Hoyles (1996) found that changes in teachers' beliefs often centred on epistemological concerns, involving a broadening of the range of activities which could be considered mathematical and connecting to changes in their personal, pedagogical and professional beliefs. Laborde (2001) used the "perturbation" of introducing Cabri-Geometre into four mathematics teachers classroom practice as a window onto their tacit hypotheses and beliefs about teaching and learning mathematics, noting that such a situation can also catalyse change in teachers' beliefs, thus restructuring their mathematics. For example, an activity proposed by the researcher "challenged the beliefs" of one of the teachers about the kind of problems he could ask his students to do and the level of conceptual reasoning and software use that he could expect from his students (Laborde, 2001). Bottino and Furinghetti (1996) outline a typology of secondary school mathematics teachers' conceptions in relation to technology and teaching mathematics with technology when confronted with a change in the mathematics curriculum requiring the inclusion of computer use in lessons.

Another set of studies documents the influence of teachers' beliefs on their integration of technology into classroom practice. For example, in their review of research on technology in mathematics education, Zbiek et al (2007) identify the constructs of *pedagogical fidelity* and *privileging* as useful in understanding the extent and nature of technology integration in a teacher's classroom practice. Pedagogical fidelity is described as the degree to which teachers' beliefs about the way a digital technology allows students to act mathematically coincide with their beliefs about the nature of mathematical learning (Zbiek et al., 2007). Privileging is a notion developed by Kendal and Stacey (2001) to describe how teachers, consciously or unconsciously, frequently use or place a priority on certain things in their practice, for example, types of representation, skills or concepts and by-hand or by-technology methods (Zbiek et al., 2007). Both these constructs relate to teachers' conceptions of mathematics as a discipline (Thompson, 1992), their beliefs about the nature of teaching and learning mathematics and how these interact with their beliefs about technology. Employing sociocultural perspectives on learning, Goos (2005; 2014) traces the development of teachers' professional identities as technology users, by focusing on their negotiation of the changing relationships between their teaching environment, actions, and beliefs.

These studies have in common a focus on teachers' global conceptions of mathematics as a discipline and on teachers' beliefs about the nature of teaching and learning mathematics with technology. In the terms of the present study then, they focus on global aspects of teachers' espoused theories and theories-in-action (Argyris and Schon, 1974, see Chapter 1). They do not tend to focus on teachers' knowledge of specific mathematical concepts in relation to technology, either in relation to their espoused theories or theories-in-action. This is an important omission since the documented shifts in teachers' views suggest a move towards models of teaching aimed at developing conceptual understanding. Such models require a great deal of knowledge for successful implementation (Thompson & Thompson, 1996) and as noted earlier, inconsistencies between teachers' professed beliefs and practices may be the result of lacking sufficient knowledge and skills necessary to implement them (Thompson, 1992).

2.6.2 Teachers' knowledge or understanding of mathematics (with technology)

Zbiek and Hollebrands (2008) cite two studies (Doerr & Zangor, 2000; Heid, Glendon, Zbiek & Edwards, 1998) which they state give general indications that teachers' knowledge of mathematics influences their use of technology in the classroom. Teacher knowledge is not the main focus of the research in either of these studies, rather it is analysed as a factor with explanatory value in understanding teachers' practices. Nevertheless these studies explicitly analyse teacher knowledge, as opposed to the studies in the previous section which focus on teacher beliefs.

Heid et al (1998) separate the influences of teachers' mathematical understanding from technology-related influences on interviewing, precluding an analysis of teachers' mathematical knowledge for teaching as a synthesis of mathematical, pedagogical and technology knowledge. Thus the technological context seems largely tangential to Heid et al's (1998) analysis of the mathematical understandings of Sara, one of the teachers participating in their study of teachers learning to do task-based interviews to assess students' understandings of algebra. They analyse Sara's confusion of parameter and variable and weaknesses in her understanding of quadratic functions and suggest that her limited understanding led her to ask misleading questions in her interviews with students. Their analysis could be seen as identifying gaps in Sara's *foundational knowledge* (Rowland et al., 2005) of algebra, although they do not refer to any research on mathematical knowledge for teaching. However, there is no indication of how the technological context impinged either on Sara's personal mathematical understandings or on her mathematical knowledge for teaching. For example, Heid et al (1998) do not discuss Sara's understanding of how situating the algebraic content within a technological context might influence her interview questioning. On the other hand, Heid et al (1998) note teacher Leanne's inattention to her students' mathematical understanding exhibited in CAS-based strategies alternative to the ones she set out to teach. Heid et al (1998) ascribe Leanne's inattention to her views of technology as curriculum – a set of skills to be learned – rather than to weaknesses in the synthesis of her mathematical, technological and pedagogical knowledge. Alternatively her failure to attend to her students' understanding could be taken to indicate a weakness in her technology-situated mathematical knowledge for teaching. In the cases of Sara and Leanne, Heid et al (1998) can be seen to focus purely on content knowledge and technological knowledge respectively, neglecting for example the dyadic and central triadic constructs of the TPACK framework.

The technological context appears similarly tangential to Yerushalmy and Chazan's (2002) discussion of teacher knowledge of school algebra. When faced with the question of helping a student to understand the meaning of the solution to $3x + 7 = 2(x + 5) + x - 1$, the teacher in their study suggested graphing $y = 3x + 7$ and $y = 2(x + 5) + x - 1$ on a two-dimensional Cartesian plane and looking for the x-

coordinate of the intersection point. She went on to recognise two tensions arising from her graphical approach that would be liable to cause her students confusion. Firstly, in the graphical representation, x is seen as a variable rather than an unknown as in the equation. Secondly, the teacher herself had difficulty in distinguishing her graphical approach to solving an equation in one variable from using graphs to solve a system of equations in two variables. Yerushalmy and Chazan (2002) suggest the teacher's confusion arose from weaknesses in her substantive mathematical knowledge, in relation to her ability to discriminate between functions and equations (Chazan et al., 1999). Like Heid et al's (1998) analysis of Sara's knowledge, Yerushalmy and Chazan (2002) indicate gaps in the teacher's foundational knowledge of algebra, but do not suggest how the technological context impinged on the teacher's understanding - other than by making the use of a graphical approach more amenable. In terms of the TPACK framework, Yerushalmy and Chazan (2002) appear to focus solely on the teacher's content knowledge and pedagogic content knowledge, to the exclusion of the technologically oriented constructs.

Doerr and Zangor (2000) provide a brief description of one teacher's knowledge and beliefs about the graphic calculator as reflected in her pedagogic strategies. Like Heid et al (1998), their description of teacher knowledge is not informed by research on mathematical knowledge for teaching. Doerr and Zangor (2000) initially emphasise the strength of the teacher's technical knowledge, evidenced by her own competency in using the calculator and her ease in answering students' questions about calculator procedures. Similarly to Heid et al (1998), Doerr and Zangor's (2000) emphasis on the teacher's technical knowledge (TK) is at the expense of considering how her pedagogic strategies reflect a synthesis of the teacher's technical, mathematical and pedagogic knowledge, as suggested by the transformative model of TPACK and by Ruthven (2011). For example, unlike Leanne, this teacher was able and willing to accommodate students' suggestions of alternative calculator strategies while using the overhead projection unit. Due both to the brevity of their description and their interpretation of this event from a purely technical point of view, Doerr and Zangor (2000) provide no indication of whether the teacher was able to fuse her technical flexibility with her mathematical and pedagogical knowledge to develop such unanticipated contributions in ways that benefited her students mathematically. Describing such a fusion might contribute to understanding mathematics teachers' *contingent knowledge* (Rowland et al., 2005) in a technological situation.

Doerr and Zangor's (2000) description of the teacher's knowledge gives some further indications of what mathematical knowledge for teaching using technology might entail. They state that the teacher drew students' attention to the mathematical limitations of the calculator and that she encouraged them to interpret calculator results to provide meaningful responses to problem situations (Doerr & Zangor, 2000). Both these pedagogic strategies suggest possible attempts to manage issues of 'double reference', where the calculator syntax diverges from the conventions of school mathematics (Ruthven, 2002). Issues of double reference tend to be invisible to mathematically expert and computationally experienced technology users (Ruthven, 2002, p. 286): thus *doing* mathematics using technology only requires an *implicit* recognition of such issues. The suggestion made here is that *teaching* mathematics using technology involves an *explicit* recognition of these issues fused with pedagogic knowledge of strategies for managing them. Doerr and Zangor (2000) do not analyse the teacher's knowledge in this way, nevertheless their description might suggest a case of teachers' knowledge-in-action, drawing out mathematical coherence by making *connections* (Rowland et al., 2005) between mathematical environments.

Bowers and Doerr (2001) make teacher knowledge the main focus of their research to the extent that they analyse growth in prospective teachers' knowing when using technology. Although they occasionally use the phrase pedagogic content knowledge, they do not reference Shulman (1986) or any research on mathematical knowledge for teaching. Bowers and Doerr (2001) separate their prospective teachers' insights into categories of mathematical and pedagogical knowing, although it is not always clear on what basis they make this distinction. For example, they categorise recognition of the *importance of appropriate contexts* for situating mathematical content as a mathematical insight. However, it could equally be seen as a pedagogic insight or, perhaps more properly, an insight combining both mathematical and pedagogic knowing. Certainly, knowledge of the most powerful analogies and illustrations of mathematical ideas is included by Shulman (1986) in his description of pedagogic content knowledge. Bowers and Doerr (2001) do not indicate whether the use of technology influenced what counted as an appropriate context, however such an insight might exemplify teachers' *transformation* (Rowland et al., 2005) of their own meanings and descriptions in a technological context to make them amenable to learners. Only one of the insights Bowers and Doerr (2001) identify appears directly linked to teaching mathematics with technology: the "pedagogic insight" of *the influence of hidden* *supports and constraints of technology on students' mathematical activities*. They describe participant Julie's understanding that the use of discontinuous velocity graphs was an artefact of the design of the *MathWorlds* software, but her regarding it as a pedagogical impediment which she would have to overcome. Other participants instead saw opportunities for understanding the value of mathematical abstraction in the discrepancies between *MathWorlds* and 'real world' data. In either case, there are similarities between this insight and the teacher's attempts to manage issues of double reference in Doerr and Zangor's study. Both involve an appreciation of the mathematical limitations of the technology fused with an understanding of their pedagogical implications and could therefore be seen as exemplifying the central TPACK construct.

2.6.3 Holistic accounts of teacher practice involving technology integration: an implicit focus on teacher knowledge

As previously indicated, the 17th ICMI Study *Mathematics Education and Technology – Rethinking the Terrain* (Hoyles & Lagrange, 2010) demonstrates the shift in research on technology in mathematics education towards a focus on the role of the teacher. However, none of the chapters in the section entitled *Teachers and Technology* explicitly focus on teachers' knowledge with respect to technology integration. Instead, teachers' knowledge emerges as an implicit focus. Indeed, in chapter 14, Goos et al (2010, p. 321) note each of the three papers under discussion "implicitly drew attention to the mathematical and pedagogical content knowledge that teachers require in order to integrate technology into their classroom practice" but do not discuss this in any further detail, relegating teacher knowledge as one amongst many other factors influencing technology integration.

Similarly, in studies aiming to give holistic accounts of teacher practice involving technology integration, teacher knowledge emerges as an implicit focus. Mapping the pedagogical opportunities provided by mathematical software, Pierce and Stacey (2010) suggest that teachers may deliberately exploit the constraints, anomalies or limitations of technology to provoke students' mathematical thinking. As Pierce and Stacey (2010) mention, this implies an appreciation of the issues surrounding the *mathematical fidelity* of the software - that is "the faithfulness of the tool in reflecting the mathematical properties, conventions, and behaviours (as would be understood or expected by the mathematical community)" (Zbiek et al., 2007). Similarly, in chapter 8 of the ICMI study, Olive et al (2010) tentatively suggest that mathematics teachers should have an understanding of the issues surrounding mathematical fidelity and be able to react to them appropriately, whilst noting that the question of "who needs to know what" about such issues has not previously been discussed in depth. The construct of mathematical fidelity corresponds in large part with the issues of double reference discussed in the previous section. Having an appreciation of the mathematical fidelity of a particular software package could be considered as part of the TCK construct – one that Bowers and Stephens (2011) neglect in their consideration of the TPACK framework. Pierce and Stacey (2010) note that their approach only addresses the (positive) pedagogical opportunities of using mathematical software, thus they do not consider potential drawbacks. Knowing pedagogic strategies both to exploit issues of mathematical fidelity or alternatively being aware of potential misconceptions arising from such issues and knowing how to deal with them could exemplify the central TPACK construct.

Monaghan (2004) also attempts to capture the "wholeness of teachers' practices" in their attempts to integrate technology. He employs Saxe's four-parameter model in an effort to produce holistic accounts of teacher practice. Teachers' *prior understandings* feature as one of the parameters in Saxe's model, incorporating teachers' beliefs about teaching and learning, the nature of mathematics and their mathematical and technical proficiency and also their understandings with regard to social practices such as how they plan their lessons. In terms of mathematical knowledge for teaching, 'prior understandings' suggests the foundational knowledge component of the Knowledge Quartet (Rowland et al., 2005), thus teachers' knowledge-in-action remains a largely implicit consideration. Following Monaghan (2004), Lagrange and Erdogan (2008) adopt Saxe's model to analyse the practices of two teachers (one a technology sceptic; the other an experienced technology user) as they use spreadsheets to teach their students about linear and exponential progressions. Whilst their description of teachers' prior understandings is brief, focusing on detailing the teachers' beliefs about technology, Lagrange and Erdogan (2008) conclude that having a good disposition towards technology is insufficient for easy integration. Contra Bowers and Stephens (2011) view of TPACK simply as having a positive stance towards technology, Lagrange and Erdogan (2008) argue that teachers need an awareness of the necessity of making students' recognise and use the spreadsheet as a mathematical tool (rather than as a tool for effecting neat presentation), the necessity of knowing how to create tasks which effectively support mathematical use of the spreadsheet and a more conscious appreciation of spreadsheet techniques both in terms of their pragmatic value in increasing mathematical productivity and in terms of their epistemic value in supporting conceptual understanding. Their analysis suggests that the introduction of spreadsheets influences what counts as an appropriate context (see discussion of Bowers and Doerr, 2001) and provides indications of how the transformation component of the Knowledge Quartet (Rowland et al., 2005) might be exemplified in a technological context.

2.6.4 Frameworks for analysing individual teacher knowledge in research on technology in mathematics education

As noted above, *The Mathematics Teacher in the Digital Era* (Clark-Wilson et al., 2014) and the special issue in the 45th volume of *ZDM – The International Journal on Mathematics Education* on *Re-sourcing Teacher Work and Interaction* (Pepin et al., 2013) bring to the fore a range of frameworks and constructs for analysing individual teachers' knowledge in relation to using technology to teach mathematics. Many of these derive from the instrumental approach, for example, the documentational approach (Gueudet et al., 2014; Gueudet & Trouche, 2009; 2011; Pepin et al., 2013), instrumental orchestration (Trouche, 2004; Drijvers et al., 2010; Drijvers et al., 2013; Drijvers et al., 2014) and teachers' personal and professional genesis (Clark-Wilson, 2014; Haspekian, 2005; 2014). As argued in Chapter 1, the centrality of the construct of instrumental/documentational genesis, i.e. the process of appropriation, means that these studies tend to focus on teachers' learning in relation to using technology to teach mathematics. Whilst teachers' learning is important, it is not the primary focus of this study. Instrumental orchestration is an exception here in that it focuses on how teachers co-ordinate students' instrumental geneses. Finally, Ruthven's (2007; 2009; 2014) Structuring Features of Classroom Practice framework presents an alternative for analysing individual teachers' knowledge to frameworks based on the instrumental approach.

Gueudet and Trouche (2009; 2011) develop the *documentational approach* as a holistic framework for studying teachers' professional development focusing on their use of resources, as an extension of the *instrumental approach* (Artigue, 2002; Guin & Trouche, 1999). The interwoven nature of mathematical knowledge and the technical skills needed for competent use of technology is highlighted by the instrumental approach (Guin & Trouche, 1999; Trouche, 2005). As Fuglestad (2011) suggests, the instrumental approach may offer a way of describing the development of TCK in terms of the process of instrumental genesis as well as a possible structure for TCK in terms of utilisation schemes. Similarly, the documentational approach may offer a way of describing the development of TPACK in terms of documentational genesis as well as a possible structure for TPACK in terms of utilisation schemes. Their inference of operational invariants such as *'a DGE can be used to help the students to observe properties of measures'* (Gueudet & Trouche, 2011) points to teachers' tacit knowledge-in-action. However, research on teacher knowledge in mathematics education does not centrally inform their framework: they note briefly Shulman's (1986) categorisation of teacher knowledge, stating that for now they consider "teachers' knowledge as a whole, without sorting out its different kinds" although this might be an area for future research and refinement of the theory (Gueudet & Trouche, 2009, p. 201). Also building on the instrumental approach, Drijvers et al (2010) use the notion of instrumental orchestration to analyse teacher practices involving technology integration. They identify archetypal teacher behaviours, such as the *Technical-demo* (Drijvers, et al., 2010) orchestration concerning the demonstration of tool techniques by the teacher, which, given their lack of specificity in relation to mathematical content, could be taken to exemplify the TPK construct. Again these orchestrations appear to focus on teachers' tacit knowledge-in-action.

Ruthven (2007; 2009) identifies five factors structuring teachers' classroom practice that together provide a framework for developing a holistic understanding of technology use in teaching mathematics, specifically *working environment*, *resource system*, *activity format*, *curriculum script* and *time economy*. Ruthven employs this five factor framework to analyse adaptations in teachers' craft knowledge (Brown & McIntyre, 1993; Leinhardt, 1988) as they attempt to integrate technology into classroom practice. Ruthven's use of terms such as craft knowledge and "expertise" (Ruthven, 2014) puts a deliberate emphasis on teachers' tacit knowledge in action. In particular, in taking a craft perspective, Ruthven et al (2009, p. 281) seek to draw away from what they term "the more decontextualised and rationalistic approach to characterising 'a professional knowledge base for teaching' (Wilson, Shulman, & Richert, 1987)". Nevertheless, the framework of factors structuring classroom practice tends also to lend a structure to teachers' craft knowledge. For example, the adaptations in craft knowledge that Ruthven et al (2009) associate with working environment appear mainly to coincide with knowledge of general pedagogic strategies situated in a technology environment (TPK) – not specific to the type of software or to mathematics teaching per se. In particular, Ruthven et al's (2009, p. 281) description of curriculum script, as interweaving the "mathematical ideas to be developed, appropriate topic-related tasks to be undertaken, suitable activity formats to be used and potential student difficulties to be anticipated… ", bears some resemblance to Shulman's (1986) strategic form of pedagogical content knowledge. Finally, Ruthven et al's (2009) assertion that elements of adaptation associated with working environment, resource system and activity format interweave in the development of teachers' curriculum scripts, together with Ruthven's (2009) identification of curriculum script with Gueudet and Trouche's (2009) concept of documentation systems, might be seen to provide support for the transformative model of TPACK.

2.7 Identifying research questions

This section summarises the conclusions drawn from the review of literature. These findings led to the identification of Research Questions 2, 2a and 2b listed in Chapter 1.

2.7.1 The nature and content of teachers' mathematical knowledge for teaching using technology

Research on teacher knowledge from the field of technology in education is immature: conceptualisations of the knowledge teachers' use in teaching with technology remain underdeveloped. This is exemplified by the theoretical and methodological difficulties surrounding the TPACK framework. A fundamental theoretical issue for the TPACK framework lies in the conflict between transformative and integrative models of the central TPACK construct. As a result, the constructs representing the intersections of technology, pedagogy and content knowledge i.e. TCK, PCK, TPK and in particular the central TPACK construct are ambiguous, thus the nature of teachers' knowledge for teaching using technology remains unresolved. Further, little work has been done to contextualise the TPACK constructs in the field of mathematics education. By comparison, research on teacher knowledge in the field of mathematics education is relatively well conceptualised, however it has rarely considered teachers' mathematical knowledge for teaching in the context of technology use. In addition, research on teacher knowledge in mathematics education has been dominated by research in primary/elementary education, although there is evidence that the issues raised might well extend to the secondary sector (Hodgen, 2011). Research on technology in mathematics education, whilst exploring the relationship between global aspects of teachers' espoused theories and theories-in-action and their technology use in some depth, has paid relatively little attention to teachers' knowledge of specific mathematical concepts in relation to technology. Research on technology in mathematics education that focuses explicitly on teacher knowledge as a factor in explaining teachers' practices, but not as a central research focus, is rarely informed either by studies of teacher knowledge in mathematics education or by studies of teacher knowledge in the field of technology in education, thus they do not build towards a systematic analysis of mathematical knowledge for teaching using technology. For example, while Heid et al (1998) consider teachers' technical and mathematical knowledge separately, Doerr and Zangor (2000) focus their description of teacher knowledge primarily on technology to the exclusion of pedagogy and content and Bowers and Doerr (2001) point to teachers' mathematical and pedagogical insights without specifying clearly how they are situated in the technological context. In none of these studies, do the researchers make clear how they distinguish these different 'types' of knowledge, nor on what basis they choose to include or exclude consideration of either one or more of mathematical, pedagogical and technical knowledge. Nevertheless, there are sporadic instances where their descriptions give indications towards viewing teachers' knowledge as a synthesis of mathematical, pedagogical and technical domains. Similarly, research aimed at producing holistic accounts of teacher practice involving technology integration tends to provide support for a transformative model of TPACK. However, since teachers' knowledge is usually an implicit focus in such studies, such indications remain sporadic and unsystematic. More recently, frameworks such as the documentational approach (Gueudet & Trouche, 2009; 2011), instrumental orchestration (Trouche, 2004; Drijvers et al., 2010) and Ruthven's (2007; 2009; 2014) Structuring Features of Classroom Practice framework have been developed, providing a means for analysing individual teachers' knowledge in relation to using technology to teach mathematics. The documentational approach has a tendency to focus on teachers' learning, which it is not the primary focus of this study. All three frameworks appear to have a tendency to highlight teachers' tacit knowledge-in-action, rather than providing a focus on teachers' articulated knowledge or espoused theories, although this is more deliberate in Ruthven's framework. As a result, the nature and content of teachers' mathematical knowledge for teaching using technology, represented by the central TPACK construct, remains an unresolved question. This conclusion led to the identification of Research Question 2:

RQ2 *What is the nature and content of teachers' mathematical knowledge for teaching using technology, as represented by the central TPACK construct?*

Research on teacher knowledge in the field of mathematics education has also highlighted the importance of a connectionist orientation (Askew et al., 1997), defined mainly in terms of knowledge for the purposes of the present study, and of connected knowledge for teaching (Ball, 1990; Ma, 1999). An intended outcome of this research will be to explore these issues in relation to technology, in particular in terms of the central TPACK construct, by contrasting connectionist and transmissionist teachers' use of digital technologies.

2.7.2 Measuring teachers' mathematical knowledge for teaching using technology

Measurements of teacher knowledge in the field of technology in education appear not to have moved beyond self-report questionnaires. A few researchers have created survey instruments to measure the constructs of the TPACK framework, however these have relied on self-reports of knowledge, criticised by Lawless and Pellegrino (2007) for reflecting teachers' confidence or self-efficacy rather than actual knowledge. Similarly, within the field of technology in mathematics education, measures of teachers' mathematical knowledge for teaching using technology have not progressed beyond self-reports of technical competence or confidence. In contrast, relatively sophisticated measures of teachers' mathematical knowledge for teaching have been developed in research on teacher knowledge within the field of mathematics education. These measures have succeeded in providing evidence linking teachers' mathematical knowledge to student achievement (Hill et al., 2005; Baumert et al., 2010) as well as evidence supporting the distinction between content and pedagogic content knowledge (Baumert et al., 2010). However, research on constructing measures of teachers' mathematical knowledge in teaching has only recently begun to explore the impact of cultural context (e.g. Delaney et al., 2008; Blomeke & Delaney, 2012). The impact of the methods with which test items are applied (e.g. multiple-choice questions, paper-and-pencil tests etc) and the mathematical topics covered on what is measured has received relatively little attention. In particular, the development of test items embedded in a technological context to measure teachers' mathematical knowledge for teaching appears to have been neglected. This area still requires substantial development, leading to the identification of subsidiary Research Question 2a:

RQ2a *How does a conceptualisation of teachers' mathematical knowledge for teaching using technology suggest ways in which such knowledge could be measured?*

Conceptualising mathematical knowledge for teaching using technology necessarily exists in a dialectical relationship with measuring such knowledge: that is, exploring one necessitates exploration of the other and so on in an iterative cycle. For example, Hill et al's (2008) failure to establish a clear distinction between KCS and subject matter knowledge using the Michigan test items, led them to recognise weaknesses in their original specification of KCS and subject matter knowledge. Hill et al (2008, p. 373) describe their paper as "a first effort to conceptualize, develop, and test measures of teachers' knowledge of content and students (KCS)". In reporting this first effort, Hill et al (2008, p. 373) ultimately connect "all three pieces of this work, tying the conceptualization directly to the specification of items, and tying results from field tests back to strengths and weaknesses of the initial conceptualization."

2.7.3 Teachers' mathematical knowledge for teaching using technology as distributed

Finally, most research on teacher knowledge, whilst recognising the situated nature of knowledge, remains underpinned by essentially individualistic assumptions (Putnam & Borko, 1997). In particular, research on teacher knowledge informed by views of cognition as *distributed* (Hutchins, 1995) across persons and technology remains underdeveloped (Putnam & Borko, 1997). In this respect, research on teacher knowledge in the field of mathematics education appears to follow a pattern similar to the wider field of education. Mishra and Koehler's criticism of the separation of PCK and curriculum knowledge could be seen as a call to investigate the *distributed* nature of mathematical knowledge in teaching. However, they stop short of this, hence in the TPACK framework knowledge remains internal to the individual.

In research on technology in mathematics education, the instrumental approach appears commensurate with Hutchins' (1995) distributed view of cognition in the sense that, by considering an instrument as the combination of an artefact with the user's schemes of use, knowledge appears distributed across both teacher and technology. However, as noted in Chapter 1, the instrumental approach has a tendency to focus on teachers' learning, which it is not the primary focus of this study. Instead, this study concentrates on teachers as workers where the focus is on the knowledge they 'have', using technology in their work of teaching mathematics. Hence Hutchins' (1995) view of distributed cognition seemed more appropriate to this study because it focuses on analysing the interaction of humans with artefacts in the workplace.

Putnam and Borko (1997; 2000) briefly describe examples of digital technologies and forms of communication between people across which they suggest knowledge for teaching could be distributed. Putnam and Borko (2000) assert that distributing knowledge for teaching across people and certain types of digital technologies might have the potential to transform practice whereas other forms of digital technologies (simply) serve to support existing practices. Crucially, in neither case do they specify *how* knowledge for teaching might be distributed across either people or the digital technologies they identify. For example, Putnam and Borko (1997, p. 1287) suggest teachers might augment their pedagogical thinking through making "judicious use" of new information technologies available via the Internet or existing technologies such as textbooks (see also Putnam & Borko, 2000). Making judicious use of such technologies suggests teachers might require a thorough, *apriori* knowledge of these technologies for the purposes of teaching. Thus it remains unclear to what extent such technology could augment teachers' thinking. In other words, it remains unclear to what extent knowledge for teaching (using these technologies) would necessarily be internal to the teacher and to what extent it could be said to reside in the technology itself i.e. what part of knowledge for teaching could be considered as being distributed across the technology. Similarly, with Hoyles and Noss' (2009) notion of *out-sourcing*, there is a sense that the user has to 'have' knowledge in the first place, before devolving it to the technology. Again in this case, it remains unclear to what extent knowledge could be considered as being distributed across the technology.

The strength of Hutchins' (1995) argument for a distributed view of cognition lies in his specification of minimum knowledge requirements for individual persons to carry out the computational tasks necessary for navigating a military ship *in interaction* with artefacts and other humans (e.g. reading bearings, pp. 137-140). By specifying the minimum knowledge requirements necessary for an individual to carry out a computational task in interaction with an artefact, Hutchins (1995) provides an indication of *how* cognition may be distributed across both human and artefact. Thus specifying minimum knowledge requirements, provides a means of indicating how and to what extent teachers' knowledge is distributed across technology. This conclusion led to the identification of subsidiary Research Question 2b:

RQ2b *To what extent is the mathematical knowledge made available through a teachers' interaction with technology distributed across the individual teacher and the technology?*

2.8 Summary

This chapter has reviewed the literature selectively for the purposes of this study. This led to the identification of the main research question relating to mathematical knowledge for teaching using technology, RQ2, and two subsidiary research questions RQ2a and RQ2b. In addition, this chapter provided a more detailed analysis of potential frameworks for analysing individual teachers' own knowledge in relation to using technology to teach mathematics. This analysis supports the selection of the TPACK framework (Mishra & Koehler, 2006) and the Knowledge Quartet (Rowland et al., 2005) in Chapter 1 as frameworks suitable for this purpose and for the purposes of this study.

Chapter 3 - Methodology

This chapter sets out how the research for this PhD study was designed to address the questions identified in the introductory chapter (see Chapter 1). The research design is justified with respect to the epistemological stance and socio-cultural perspective in relation to technology use outlined in Chapter 1. The methods for data collection and analysis are also described and justified.

The focus of this study is on individual teachers' knowledge and how it is involved in interacting with technology to produce the mathematical knowledge made available in the classroom. In Chapter 1, it was argued that there is an imperative for exploring a connectionist orientation (Askew et al., 1997) in relation to technology use. In the terms of this study, a connectionist orientation is defined mainly in terms of knowledge. Hence comparing and contrasting connectionist-oriented teachers' use of technology to that of transmissionist-oriented teachers provided a means of making mathematical knowledge for teaching using technology more visible.

This study adopted a mixed-methods approach towards investigating individual teachers' knowledge and how it is involved in interacting with technology to produce the mathematical knowledge made available in the classroom. Adopting a mixed methods approach represented a pragmatic methodological means of addressing the research questions identified and justified in Chapters 1 and 2. The quantitative and qualitative elements of this study were of similar importance (Johnson & Onwuegbuzie, 2004) in the research design. For ease of reference, the research questions are listed in the following section, below Table 3.1.

A largely quantitative approach was taken to addressing Research Question 1, using survey data to explore associations between a connectionist orientation and ICT use, in order to be able to generalise more effectively to the population of English secondary school teachers. The study did not aim for a statistically representative sample. Instead case selection was theoretically purposive (Miles & Huberman, 1994), aiming to maximise the proportion of connectionist teachers included in the sample, and to enable the collection of data on *contextual features* (Stein et al., 2007) in line with the socio-cultural perspective outlined in Chapter 1. However, the sample size of 183 teachers was considered sufficient for the purposes of statistical analysis, in line with Pampaka et al's study (2012), and in this sense it allowed a more effective generalisation from a larger number of critical cases. This quantitative approach was made possible by Pampaka et al's (2012) development and validation of a set of items that could be used to create a scale of transmissionist self-reported pedagogic practice using Rasch analysis. The scale of transmissionist self-reported pedagogic practice provided a means of selecting two connectionist and two transmissionist teachers as critical cases for investigating mathematical knowledge for teaching using technology.

Addressing Research Question 2 and the subsidiary questions 2a and 2b, a qualitative approach was judged necessary to investigate individual teachers' knowledge and how it is involved in the participatory relationship (Remillard, 2005) between teacher and technology. More specifically, a qualitative approach was judged necessary to investigate the nature and content of individual teachers' knowledge in relation to teaching using technology and how and to what extent knowledge is distributed (Hutchins, 1995) across teacher and technology. In particular, in line with the epistemological position outlined in Chapter 1, it was appropriate both to probe teachers' theories-in-action and triangulate (Lincoln & Guba, 1985) this with data on their espoused theories to provide evidence indicating mathematical knowledge for teaching using technology. Hence data collection entailed observing lessons involving ICT, followed by a post-observation interview, and a semi-structured interview or 'think aloud' based around manipulating a GeoGebra (2008) file on circle theorems. Analysing this data to investigate individual teachers' knowledge using the Knowledge Quartet (Rowland et al., 2005) provided a means of conceptualising mathematical knowledge for teaching using technology as a pre-requisite for considering how such knowledge could be measured, to address RQ2a. In analysing individual teacher's knowledge, the purpose is to make inferences about the nature and content of teachers' knowledge in general; it is not to critique the knowledge of individual teachers.

The issues raised in this introductory section will be discussed and justified further in the following section, which sets out the research design.

3.1 Research Design

The PhD study comprised two phases of data collection. The first phase of data collection was quantitative, surveying secondary school mathematics teachers' technology use in England. The second, mainly qualitative, phase of data collection used the survey data to identify four case study teachers. In this second phase, for each case study teacher, data collection involved observations of two lessons involving ICT, followed by a post-observation interview, and a semi-structured interview based around a GeoGebra file on circle theorems. Table 3.1 depicts the research design, showing which data were used to inform the research questions identified in Chapter 1. The research questions are listed below Table 3.1 for ease of reference. In addition, Table 3.1 provides an indication of how the data were analysed to address these research questions.

A survey was used to address RQ1 since the aim was to explore associations between a connectionist orientation and ICT use that could be generalised effectively to the population of English secondary school teachers. The survey data consisted of teachers' self-reports of their classroom practice. The theoretical perspective outlined in the introduction to the project (see Chapter 1) highlights the limitations of such data as a descriptive measure of teachers' actual classroom practice. In this sense, choosing a survey method represented a necessary trade-off between being able to make effective generalisations and potentially obtaining more detailed and accurate information (Hammersley, 1992, p. 186) through direct observation of teachers' classroom practice.

Table 3.1 Research Design

<table>
<tr><th></th><th>Data</th><th>Respondents</th><th>Analysis</th><th>Findings</th></tr>
<tr><td rowspan="2">RQ1</td><td>Survey data</td><td>183 respondents</td><td>Rasch and other statistical analyses</td><td rowspan="2">Chapter 4</td></tr>
<tr><td>Two lesson observations + post-observation interviews</td><td>4 respondents</td><td>Triangulation of survey findings</td></tr>
<tr><td rowspan="2">RQ2</td><td>GeoGebra interview</td><td>4 respondents</td><td rowspan="2">TPACK framework</td><td rowspan="2">Chapter 5</td></tr>
<tr><td>Two lesson observations + post-observation interviews</td><td>4 respondents</td></tr>
<tr><td>RQ2 + RQ2a</td><td>GeoGebra interview</td><td>4 respondents</td><td>Knowledge Quartet</td><td>Chapter 6</td></tr>
<tr><td rowspan="2">RQ2b</td><td>GeoGebra interview</td><td>4 respondents</td><td rowspan="2">Framework for identifying instances of distributed cognition</td><td rowspan="2">Chapter 7</td></tr>
<tr><td>Two lesson observations + post-observation interviews</td><td>4 respondents</td></tr>
</table>

RQ1 *How is a connectionist orientation towards teaching mathematics associated with teachers' frequency of use of ICT, their orientation towards ICT and their pedagogic practices involving ICT?*

RQ2 *What is the nature and content of teachers' mathematical knowledge for teaching using technology, as represented by the central TPACK construct?*

RQ2a *In what ways can teachers' mathematical knowledge for teaching using technology be measured?*

RQ2b *To what extent is the mathematical knowledge made available through a teachers' interaction with technology distributed across the individual teacher and the technology?*

Data from lesson observations of the four case study teachers presented a means of ameliorating the limitations of the survey approach by triangulating (Lincoln & Guba, 1985) the survey findings. In particular, the lesson observations of the four case study teachers provided critical cases (Miles & Huberman, 1994) with which to triangulate the survey findings of dominant practices in ICT use (see Chapter 4). They were critical cases in two senses. Firstly, the case study teachers represented critical cases of connectionist and transmissionist orientation in relation to ICT use. Using the survey data, the four case study teachers were chosen to be two of the most connectionist-oriented and two of the most transmission-oriented teachers from the survey respondents who had volunteered to be contacted for this purpose. The case study teachers were also chosen so that one of the connectionist-oriented teachers came from a school supportive of ICT use and one from a non-supportive school (and similarly for the transmissionist teachers), again identified via the survey data, in accordance with the socio-cultural perspective towards teachers' ICT use described in Chapter 1. In addition, the four case study teachers had described themselves as being confident with ICT. Thus they were likely to be technology enthusiasts who would represent strongly transmissionist or connectionist-oriented practice involving ICT if any association were present.

Secondly, the case study teachers appeared to view the lesson observations as an opportunity to 'show-case' their best or innovatory practice involving ICT. The only stipulation the case study teachers were given with regard to the ICT lessons was that one should take place in a whole-class context with an IWB and the other in a computer suite, where the pupils had direct access to the hardware and software. This stipulation was an attempt to recognise the constraints placed on teachers' classroom practice due to different *working environments* (Ruthven, 2007; 2009), also reflected in the structure of the survey questions. No other stipulations were made in terms of the observed lessons to avoid over-burdening the teachers for ethical reasons and for practical reasons

– they were volunteers after all. Nevertheless, the lessons observed appeared to represent the teachers' attempts (not always entirely successful) to demonstrate what they regarded as ideal, innovatory practice, rather than their standard or normal practice involving ICT. Again, in this sense, the observed lessons were critical rather than 'typical' cases of the case study teachers' classroom practice involving ICT. Thus the lessons observed were likely to represent strongly transmissionist or connectionist-oriented practice involving ICT if any association were present.

In terms of RQs 2, 2a and 2b, the case study teachers again represent critical cases for investigating mathematical knowledge for teaching using technology. For reasons described above, the case study teachers were likely to be technology enthusiasts and therefore more likely to have acquired some mathematical knowledge for teaching using technology, for example through their experience of using technology in their own classroom practice. It was also critical that between the teachers there should be variation in dimensions that might be associated with mathematical knowledge for teaching using technology. Thus contrasting the case study teachers' use of technology would be more likely to force productive reflection, serving to make more "visible" mathematical knowledge for teaching using technology, to highlight the absences (things left unsaid) as well as the presences (things made explicit), (Hoyles et al., 1999; Venkatakrishnan, 2004). The choice of two connectionist-oriented and two transmissionist-oriented teachers and the level of school support provided two such dimensions of variation. Within the group of four case study teachers there was also variation in terms of teaching experience and expertise with technology prior to teaching (e.g. from a previous career or university degree). In addition, and in relation to RQ2a, the choice of two connectionist-oriented and two transmissionist-oriented teachers enabled an investigation of individual teachers' knowledge using the Knowledge Quartet (Rowland et al., 2005) to conceptualise mathematical knowledge for teaching using technology as a pre-requisite for considering how such knowledge could be measured.

Since no stipulation had been made regarding the lesson observations, other than that one should take place in a computer suite and the other in a whole-class context with an IWB, comparability was problematic e.g. the lesson content and technology used was different across lessons. Thus the semi-structured interviews based around a GeoGebra file on circle theorems were judged necessary to provide a common situation across which the case study teachers' use of technology for teaching mathematics could be contrasted as discussed above. The case study teachers were prompted to show and discuss how they would use the diagrams presented in the GeoGebra file to demonstrate the angle at the centre theorem to their pupils. Circle theorems were chosen since it is a topic, in the English mathematics curriculum, which is commonly identified with the use of dynamic geometry software (Ruthven et al., 2008). It was therefore reasonable to assume that the case study teachers would be familiar with technological resources similar to the diagrams presented in the GeoGebra file and might even have previously used such resources in their own teaching. Thus they would be likely to have some mathematical knowledge for teaching circle theorems using the GeoGebra file, even if they were unfamiliar with the particular software. In addition, the topic of circle theorems is at the apex of geometry in the compulsory English mathematics curriculum, since it is typically where proof is introduced. Hence it provided a potentially challenging context even for experienced teachers who were both mathematically and technologically confident. In addition, much of the research conceptualising mathematics teacher knowledge has been limited in terms of its curriculum coverage, for example, focusing on arithmetic operations (Ma, 1999) and on numeracy (Askew et al., 1997) – see Chapter 2. Hence conceptualising teacher knowledge in relation to a topic in geometry contributes to widening the evidence base for mathematical knowledge in teaching.

In line with the epistemological position outlined in Chapter 1, it was appropriate both to observe the case study teachers' theories-in-action and triangulate (Lincoln & Guba, 1985) this with data on their espoused theories to provide evidence indicating mathematical knowledge for teaching using technology. In addition, the term "mathematical knowledge for teaching using technology" is intended to suggest a situated view of knowledge, implying that it is knowledge used in situations involving "the work of teaching mathematics using technology" (see Hill et al., 2005). Most obviously, the IWB and computer suite lesson observations provided opportunities to observe the case study teachers' theories-in-action in a situation involving the work of teaching mathematics with technology. The post-observation interviews then provided an opportunity to infer the case study teachers' espoused theories and hence triangulate them against their theories-in-action observed in the lesson, to provide evidence indicating mathematical knowledge for teaching using technology. Due to the author's past experience as a teacher of mathematics, these post-observation interviews could be reasonably construed as a discussion between colleagues, reflecting on a mathematics lesson involving technology i.e. a situation involving the work of teaching mathematics with technology. Similarly, the semi-structured interviews based around a GeoGebra file on circle theorems could be construed as a discussion between colleagues on, for example, planning a possible mathematics lesson involving technology. The GeoGebra interviews provided an opportunity to infer the case study teachers' espoused theories in relation to teaching circle theorems using technology. Since the case study teachers were also asked to manipulate the GeoGebra file in a simulation of what they might actually demonstrate to pupils, this also arguably provided an opportunity to observe their theories-in-action.

Table 3.2 Methodological strengths and weaknesses of GeoGebra interviews and lesson observations coupled with the post-observation interviews.

	+ Reliability	+ Validity
- Reliability		Lesson observations + post-observation interviews
- Validity	GeoGebra interviews	

The methodological strengths and weaknesses of the data obtained from the GeoGebra interviews and the lesson observations coupled with post-observation interviews, in terms of their validity and reliability, are summarised in Table 3.2. The extent to which the teachers' manipulation of the GeoGebra file can be argued to correspond with their theories-in-action (i.e. what they would actually do in classroom practice) represents a methodological limitation of the GeoGebra interviews. The case study teachers' own, unsolicited assertions that their manipulation of the GeoGebra file in the interview was similar to their previous classroom use of such resources provides some evidence to support this validity claim. The lesson observations and post-observation interviews ameliorate this limitation to some extent, by providing supporting evidence that claims about mathematical knowledge for teaching using technology made on the basis of the GeoGebra interview data do hold when applied in the context of actual classroom practice (see for example, the analysis in Chapters 5 and 7).

The evidence presented by this study for mathematical knowledge for teaching using technology is not really the degree to which the case study teachers' espoused theories actually correspond to their theories-in-action. Instead, of course, it is the degree to which the author perceives a correspondence between the case study teachers' espoused theory and theories-in-action. In this sense, the GeoGebra interviews may provide better opportunities for the author to perceive reliably a correspondence or non-correspondence between the case study teachers' espoused theories and their theories-in-action than the lesson observation and post-observation interview. This is because in the GeoGebra interviews, the case study teachers explain their actions in the moment, i.e. in a 'think-aloud' (Ericsson & Simon, 1993; Fox et al., 2011), in response to a pre-prepared set of interview questions. By contrast, the post-observation interviews may suffer from hindsight-bias or after-the-fact-rationalisation (Evans, 2005) by not only the case study teacher but the author as well, since the interview schedule was minimally structured to allow the author flexibility to ask questions addressing issues raised during the lesson observation.

3.2 Data collection

This section begins by describing and justifying the data collection methods employed in the quantitative phase of the project. In particular, the design and development of the survey instrument and the survey sample are discussed and justified. The case study teachers are then introduced through a brief description, indicating how each individual varied along dimensions that might be associated with mathematical knowledge for teaching using technology. Finally, the data collection methods employed during the GeoGebra interviews, lesson observations and post-observation interviews are described.

3.2.1 The survey instrument and sample

The aim of the survey was to explore associations between a connectionist orientation and English secondary school teachers' ICT use to address RQ1. Underlying this aim is the recognition that individual teachers interpret and make use of particular technologies in their classroom practice in ways that are not (fully) determined by the design and nature of the hardware and software (Remillard, 2005; Spillane, 2006). In particular, this survey attempted to take into account individual level factors and *contextual features* (Stein et al., 2007) that might influence the participatory relationship between teachers and technology. This theoretical perspective is set out in more detail in Chapter 1. The final survey instrument contained mainly closed Likert-type response formats grouped under the following sections (see Appendix A):

- A *ICT in your school* – items on access to hardware/software and school/departmental level factors effecting ICT use;
- B *ICT use in your own mathematics teaching*
  - i *Your use of hardware* - perceived impact and frequency of use of hardware;
  - ii *Using an interactive whiteboard or data projector in maths lessons* – items on frequency of software use, individual factors affecting ICT use and pedagogic practices with an IWB or data projector in a whole-class context;
  - iii *Maths lessons in a computer suite or using laptops* – similarly, items on frequency of software use, individual factors effecting ICT use and pedagogic practices with ICT in the context of a computer suite or using laptops;
- C *Your own mathematics teaching in general* – Pampaka et al's (2012) items relating to pedagogic practices in general (not specific to ICT use); and
- D *About You* – personal background details.

The survey questions in section A aimed to gather data on *contextual features* (Stein et al., 2007), such as school and departmental support for ICT, which might influence the participatory relationship between teachers and technology. Section B collected self-report data on the frequency of mathematics teachers' use of hardware and software and their pedagogic practices involving ICT. Again an attempt was made to take contextual features into account, specifically the *working environment* (Ruthven, 2009), by dividing questions between using software in a whole-class context with an IWB and using ICT in a computer suite where students have direct access to the software. The inclusion of questions on pedagogic practices involving ICT was in recognition that the design and nature of the hardware and software that teachers use does not fully determine their classroom practice. In addition, Sections B and D included questions on individual level factors, recognising that these might also influence the participatory relationship between teachers and technology.

Section C of the survey instrument comprised Pampaka et al's (2012) items relating to pedagogic practices in teaching mathematics in general i.e. not specific to ICT use. These items were included to construct a measure of teachers' transmissionist self-reported pedagogic practice as an indicator of their transmissionist or student-centred/ connectionist orientation, in line with the aim of the survey to address RQ1. Pampaka et al (2012) revised and validated Swan's (2006) set of self-report items on pedagogic practice as a means of developing a measure of transmissionist self-reported pedagogic practice in post-compulsory, 'advanced' level mathematics college classrooms. The item-stems described classroom activities associated with student-centred or teacher-centred practices. Teachers were asked to indicate the frequency with which these activities occurred in their classroom practice on a five-point scale: almost never, occasionally, about half the time, most of the time, almost always. The full set of items in listed in Appendix A and re-produced in Chapter 4 for ease of interpretation. Pampaka et al's (2012) items relate to teachers' classroom practices rather than to the knowledge that underpins a connectionist or transmissionist orientation. Nevertheless, the teachers' responses to these items provide an indicator of their espoused theories (Argyris & Schon, 1974; Pampaka et al., 2012) of teaching practice, hence they may be taken as an indirect indication of their mathematical knowledge for teaching.

The list of software used in the survey was derived mainly from Hyde's (2004) list, checked against a survey of software use by the Fischer Family Trust (2003), to ensure that the main types of software used by secondary mathematics teachers in England had been included. Notably, IWB software and the MyMaths.co.uk website (Oxford University Press, 2012) were also included in the survey. IWB technology and resources such as the *MyMaths* website have tended to be dismissed within the mathematics education community (e.g. Zbiek et al., 2007) because there is a tendency to assume they facilitate and even encourage teacher-centred practices. Indeed, they are sometimes not even regarded as 'using ICT' as a result. However, the theoretical perspective outlined in Chapter 1 suggests that teachers may use such resources with *interpretative flexibility* (Ruthven, 2008; 2009), in ways that may not appear to correspond with the outward design and nature of the software or hardware. In addition, as discussed in the following paragraph, there was evidence to suggest that these software are widely used in practice, thus omitting them would lead to a misrepresentation of teachers' ICT use.

IWB software refers to often presentation-type software that is designed specifically for use with IWB hardware, for example SMART Notebook (2014) or Promethean ActivInspire (2011). The growing presence of IWBs in mathematics lessons in England, indicated by the pilot study and other reports (e.g. Moss et al., 2007), suggests that IWB software may be used regularly by mathematics teachers and it was therefore included in the list of software for this survey. The *MyMaths* website was included since this site was known anecdotally to be widely used in UK schools (see for example, the school case studies reported in Clark-Wilson, 2008, p. 103-4). It is a subscription site offering pre-planned lessons, on-line homework and many other resources. The lessons and homework are linked to an "Assessment Management system", allowing teachers to track individual student's progress.

The survey instrument was progressively developed over the course of various phases of piloting. The initial questionnaire design was informed by previous surveys of mathematics teachers' use of ICT, primarily Hyde's (2004) survey of mathematics teachers in Southampton and Forgasz's (2002) survey of mathematics teachers in Victoria, Australia. This questionnaire was trialled with students on the Post-Graduate Certificate of Education (PGCE) mathematics course at King's College London, before being piloted with 27 schools working in partnership with King's College London to offer initial teacher education in secondary mathematics. The results of the pilot survey are reported in Bretscher (2011). As a result of this piloting, the questionnaire was re-developed to include items relating to teachers' pedagogic practices with ICT and to highlight more clearly the division of questions between using ICT in a whole-class context and using ICT in the context of a computer suite or using laptops. Items relating to school and individual factors affecting teachers' use of ICT were also re-written to aid clarity. The re-designed questionnaire was trialled in two think-alouds (Willis, 2005) with PGCE students and with three experienced in-service teachers, who completed the questionnaire and then gave verbal feedback. The theoretical perspective outlined above implies that survey respondents engage in a participatory relationship with the text of the questionnaire, actively interpreting questionnaire items in the light of their own circumstances, whilst the questionnaire items may also shape respondents' perception of these circumstances. Indeed, one of the three experienced in-service teachers, with whom the questionnaire was trialled, commented with surprise on how she perceived shifts in her own conception of what 'ICT use' meant as she progressed through different sections of the questionnaire.

Questionnaires were sent to teachers in mathematics departments in 87 secondary schools selected mainly through contacts with teacher educators in three English universities. The study did not aim for a statistically representative sample, which in any case was not feasible within the limited resources of the project. Instead, the sample was purposively directed at school level, with schools selected mainly through contacts with teacher educators in three English universities. This selection was made to ensure sufficient representation of connectionist teachers in the sample. Askew et al (1997) suggested that transmission orientations are likely to be more common amongst secondary teachers than primary teachers. Indeed, the sample of teachers in Pampaka et al's (2012) study was skewed towards a transmission orientation. By selecting schools with contacts to universities through initial teacher education programmes, this study aimed to ensure a sufficient number of connectionist-oriented teachers in the sample. In addition, purposively directing the sample at school level enabled the collection of data on *contextual features* (Stein et al., 2007) such as local school or departmental level factors effecting ICT use. This ensured that the participating schools varied across a number of characteristics that might affect access to and use of technology. For example, in London there was a successful policy drive to equip one core subject area with IWBs (defined as Maths, Science or English, see Moss et al., 2007), thus access to particular types of technology may vary according to geographical area. The sampling method ensured some geographical variation, with schools situated mainly within three areas: Greater London, West Yorkshire and the South of England (taken as comprising the counties of Hampshire, West Sussex and Dorset). The participating schools also varied across a number of characteristics that might effect access to and use of technology, including a wide range of attainment in national tests; most were state schools but some were private schools; some have speciality status and some do not; some are single sex and some are selective.

Nine questionnaires were sent to each school and 50 schools agreed to take part. A total of 188 completed individual teacher questionnaires were returned, an average of 3.8 questionnaires per school. Of these, data from 183 teachers were entered for statistical analysis; 5 were removed due to missing or problematic data in Section C of the survey. The sample size was considered sufficient for the purposes of statistical analysis, in line with Pampaka et al's study (2012). Twelve schools returned only one completed questionnaire, whilst one returned all nine. The participating teachers (101 F; 86 M; 1 unspecified) had a mean age of 38.5 years and mean length of service of 10.5 years, ranging from NQT to over 25 years experience. The majority of respondents (96) described their main responsibility as classroom teacher. The sample also included 24 heads of department, 18 deputy heads of department and 24 Key Stage coordinators. Thus whilst the survey sample cannot be said to be statistically representative of the population of English secondary school mathematics teachers, the respondents vary across a range of individual level factors that might be associated with technology use. In addition, the data may be skewed due to the clustering of teachers in schools. There may also be a potential bias in the survey sample towards teachers who are relatively well-disposed towards ICT or those wishing to be seen as frequent users of ICT. Comparing themselves to their colleagues in the maths department, only 9.0% of survey respondents thought they use ICT less or much less frequently whereas 33.5% thought they use ICT more or much more frequently. As argued in the previous section, this potential bias may be seen as a strength in relation to the selection of case study teachers as critical cases of connectionist and transmissionist orientation in relation to ICT use.

3.2.2 Introducing the case study teachers

As discussed in the previous section detailing the research design, the four case study teachers (Robert, Anne, Edward and Michael) were chosen along two dimensions of variation that might be associated with mathematical knowledge for teaching using technology, as depicted in Table 3.3. Thus the case study teachers were chosen to be two of the most connectionist-oriented and two of the most transmissionist-oriented teachers of those who volunteered. The level of school support provided the other dimension of variation. This sub-section provides a brief description of each case study teacher in relation to these two dimensions of variation and other relevant dimensions, such as teaching experience and expertise with technology prior to teaching.

Table 3.3 Choice of case study teachers along two dimensions of variation

	+ connectionist-oriented	+ transmission-oriented
low school support	Anne	Edward
high school support	Robert	Michael

Robert

Robert was one of the most connectionist teachers in the survey sample (see Chapter 4, Figure 4.1, transmissionist measure -1.01). His school was generally supportive of ICT use compared to the other schools surveyed, based on the reports of 3 respondents including Robert's own survey response. He had 4-6 years of teaching experience and held a management position within the mathematics department, as second-in-command. His initial teacher education had been via a Post-Graduate Certificate of Education and he had subsequently completed a Masters in Education degree. Robert was the most technologically proficient of the four case study teachers: his undergraduate degree was a Bachelor of Engineering in Computing. This was underlined by his use of a pupil-response system in the IWB lesson observation that he had designed and programmed with help from colleagues.

Anne

Anne was one of the more connectionist-oriented teachers in the sample (see Chapter 4, Figure 4.1, transmissionist measure -0.50). Her school was less supportive of ICT use compared to the other schools surveyed, based on the reports of 4 respondents including Anne's own survey response. Anne was the most experienced teacher of the four case study teachers, with 10-15 years of teaching experience. She held a management position as a Head of Year, responsible for the pastoral care of students (rather than a subject-related role). Her initial teacher training had been via an undergraduate degree, Bachelor of Education in secondary mathematics. She was the least technologically proficient and confident of the four case study teachers. For example, she mentioned her reliance on her son's skills with technology to help her use software to design ICT resources and on a teaching colleague to help her negotiate the filing system on her computer.

Edward

Edward was one of the more transmission-oriented teachers in the survey sample (see Chapter 4, Figure 4.1, transmissionist measure 0.74). His school was unsupportive of ICT use compared to the other schools surveyed, based on the reports of 7 respondents including Edward's own survey response. Out of the four case study teachers, Edward was the only one working in a private (i.e. fee-paying) school – the other three taught in state-funded secondary schools. His department was somewhat unusual in that the classrooms were not equipped with IWBs, having data projectors only installed instead. Edward preferred this arrangement in any case, however he did have access to IWBs by booking classrooms used by other subject departments. He had 2-3 years teaching experience and had completed a Post-Graduate Certificate of Education. He obtained a first-class grade in his undergraduate degree in mathematics and, in this sense, had the strongest mathematical background of the four case study teachers. He appeared technologically proficient and confident. For example, he mentioned obtaining permission to up-date the web-browsing software himself on school computers.

Michael

Michael was one of the most transmissionist teachers in the sample (see Chapter 4, Figure 4.1, transmissionist measure +1.01). His school was supportive of ICT use compared to the other schools surveyed, based on the reports of 6 respondents including Michael's own survey response. He had 2-3 years teaching experience and had completed a Post-Graduate Certificate of Education. His undergraduate degree was in Economics and he had subsequently completed a Masters in Economics. Michael appeared to be the least confident of the teachers in relation to his own subject knowledge. For example, he indicated that he hoped to teach economics and seemed to view teaching mathematics as a means to that end. In addition, Michael tended to treat the author as a senior colleague, commenting off-record on the similarity between conversations with his mentor in initial teaching training.

3.2.3 The GeoGebra file on circle theorems and interview protocol

The GeoGebra file comprised three diagrams relating to the circle theorem stating that angle at the centre of the circle, subtended by an arc, is double the angle at the circumference subtended by the same arc (see Figure 3.1 and Appendix N). Table 3.4 provides a list of the semi-structured interviews based around the GeoGebra file.

Table 3.4 List of GeoGebra interviews

Case study teacher	Date	GeoGebra interview
Robert	13.6.2012	Rob-GGb-int
Anne	28.6.2012 29.6.2012	Anne-GGb-intA Anne-GGb-intB
Edward	20.6.2012	Ed-GGb-int
Michael	31.5.2012	Mic-GGb-int

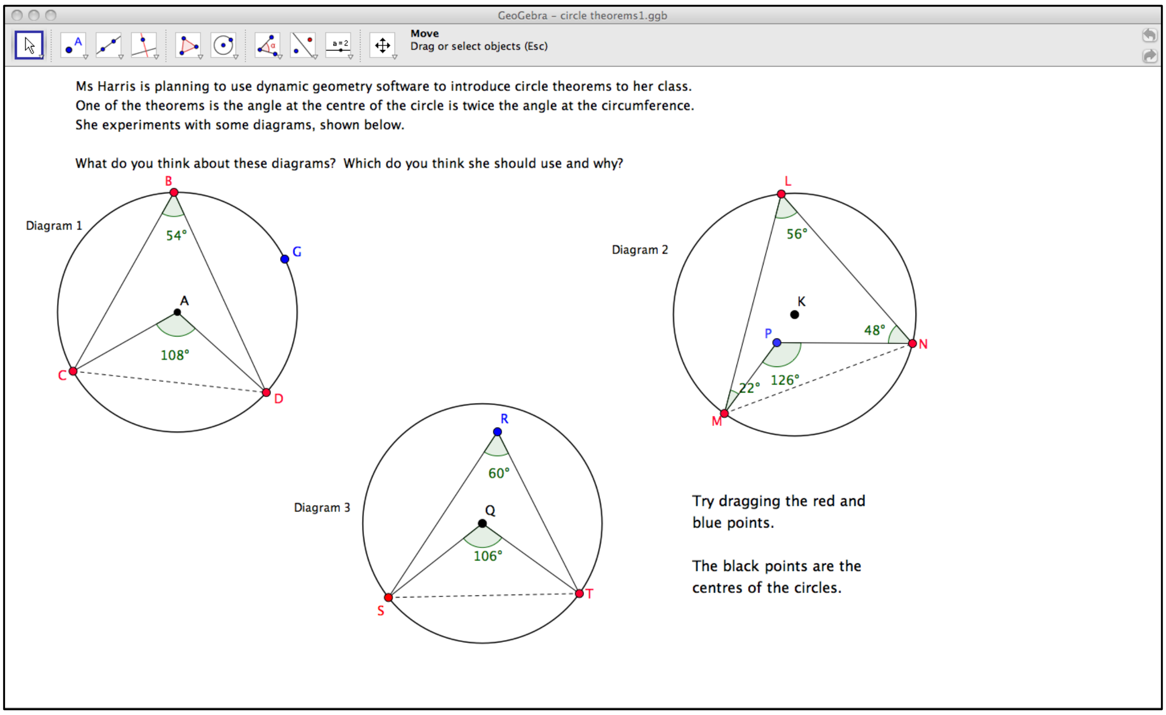


Figure 3.1 The GeoGebra interview file on circle theorems

The GeoGebra file also incorporated some text, setting the task of manipulating the diagrams in the pedagogical context of planning how to introduce pupils to this circle theorem based on a demonstration using these diagrams. The first diagram D1 was designed to be similar to resources found on a web-search. Thus the case study teachers were likely to have at least some familiarity with a dynamic diagram like D1 and possibly have even used something similar in their own lessons. The second and third diagrams were designed to be unusual by comparison: Dietmar Kuchemann's (2003) article "*Angle at the centre: taking a point for a walk*" was instrumental in their design. These diagrams could be manipulated to produce a soft construction (Laborde, 2005) of the angle at the centre theorem but could also be disrupted to produce non-examples of the theorem. These diagrams were included to provide a context that would challenge the subject knowledge of and provide some interest to a technologically proficient and mathematically confident secondary mathematics teacher. The GeoGebra interviews were semi-structured around a protocol with questions designed to cover each of the dyadic and triadic categories of the TPACK framework (see Appendix B). Before opening the GeoGebra file on circle theorems, the case study teachers were asked to practise ‘thinking-aloud’ whilst manipulating a GeoGebra file with a soft and robust construction (Laborde, 2005) of a square. The semi-structuring of the interview allowed the author some flexibility to respond to events during the interview, whilst maintaining an overall structure that would allow for and facilitate comparison. The GeoGebra interviews generally took place in a mathematics classroom at the case study teacher’s school that was not being used for teaching at that time. The author’s laptop with mouse attached was arranged on a desk so that both the author and the case study teacher could comfortably see the screen and use the mouse to manipulate D1, enabling collaboration on the task. Both the visual and audio aspects of the GeoGebra interviews were recorded on the author’s laptop using *iShowU* (Shinywhitebox Ltd, 2011) software.

3.2.4 Lesson observations and post-observation interviews

Each case study teacher was observed teaching two lessons using ICT. With the exception of Anne, one of these lessons was where the teacher used an IWB in a whole class context and the other lesson took place in a computer suite where the pupils had direct access to technology. This was a deliberate choice by the author to provide a contrast in working environment (Ruthven, 2009) that might make differences in knowledge between connectionist and transmissionist-oriented teachers more visible. Anne chose to do both her lessons in a computer suite. Since Anne was less confident using technology, the author did not insist on observing an IWB lesson for pragmatic and ethical reasons. Table 3.5 provides a list of lesson observations and post-observation interviews.

Table 3.5 List of lesson observations

Case study teacher	Observation type	Date	Post-observation Interview	Year group, mathematical topic, main use of technology
Robert	IWB	13.6.2012	Rob-IWB-int	Year 12 the Chain Rule using PowerPoint and pupil response system designed by Robert
	Computer suite	13.6.2012	Rob-CS-int	Year 8 reflection of 2D shapes using maze activities designed in GeoGebra
Anne	Computer suite 1	28.6.2012	Anne-CS1-int	Year 8 transformation of graphs using Autograph
	Computer suite 2	29.6.2012	Anne-CS2-int	Year 8 transformation of graphs using Autograph
Edward	IWB	20.6.2012	Ed-IWB-int	Year 12 graphing inverse functions using GeoGebra
	Computer suite	27.6.2012	Ed-CS-Int	Year 9 revising equation of straight-line graphs using Internet game
Michael	IWB	29.5.2012	Mic-IWB-int	Year 7 translation of 2D-shapes using pupil response system
	Computer suite	31.5.2012	Mic-CS-int	Year 8 volume of cuboid using MyMaths.co.uk

During the lesson observations, the author assumed the role of participant-observer (Cohen, Manion, & Morrison, 2000) in that she was not directly involved in the lesson, beyond brief interactions with a handful of students. The case study teacher's mathematical delivery of the lesson was the main focus of observation. Field notes were taken every three or four minutes and a descriptive synopsis (Rowland et al., 2005) of the lesson was written from memory shortly after the lesson, these were later typed up – see Appendix I. Copies of resources used in the lesson, such as PowerPoint slides, GeoGebra files, worksheets or links to online resources, were gathered when available. The post-observation interviews took place as soon as possible after the lesson observation and on the same day. The interviews were minimally structured to provide the author with flexibility to follow up and seek clarification on events/issues arising from the lesson observation. An interview schedule, included in Appendix C, was more like a list of prompts, reminding the author of areas she wished to cover in the interview, as well as providing a means to stimulate conversation when necessary. The post-observation interviews were digitally audio-recorded.

3.3 Data Analysis

This section begins by describing and justifying the statistical analyses carried out on the survey data to address RQ1 in the quantitative phase of the project. The chapter concludes with a description and justification of the qualitative analysis of data from the GeoGebra interviews and lesson observations coupled with post-observations interviews used to address RQs 2, 2a and 2b, cross-referenced with the relevant data analysis chapters where appropriate.

3.3.1 Statistical analysis of survey data

The statistical analysis presented in this thesis focuses on the use of Rasch modelling to construct a scale of transmissionist self-reported pedagogic practice from teachers' responses to Pampaka et al's (2012) items, comprising section C of the survey instrument. The Rasch model is discussed in the next paragraph and also in Chapter 4, where the results of the measure construction are presented. The Rasch analysis was carried out using the *Winsteps* (2011) software. Finally statistical analyses exploring the association between the transmissionist self-reported pedagogic practice and other variables, carried out using the *PASW Statistics 18.0* (2009) software, are described.

The Rasch Model

The Rasch model was selected because it provides a method of constructing an interval level scale of measurement, necessary for conducting parametric tests, from ordinal level data. In particular, following Pampaka et al (2012), Rasch modelling was used in this study to construct a scale of transmissionist self-reported pedagogic practice for secondary school mathematics teachers using their items. Due to its purpose for constructing scales adequate for measurement, Rasch differs from other statistical models in that the data must fit the model, rather than the other way round. That is, if the data do not fit the model then they are inadequate for the purpose of constructing an interval scale of measurement i.e. the data rather than the model is at fault and must be rejected. Coe (2008) suggests this unconventional approach may be the reason why the use of Rasch has been controversial, particularly in the UK, though in other parts of the world it is widely accepted. Similarly, Pampaka et al (2012) note the controversy over Rasch modelling, asserting its widespread use internationally in educational assessment and beyond in the field of health sciences.

Rasch modelling assumes the 'ability' of persons and the 'difficulty' of items can be measured on the same scale. In this study, the difficulty of an item can be taken to mean the tendency for the pedagogic practice in the item-stem to be endorsed. Person ability can be taken to mean the extent to which a teacher feels able to endorse a transmissionist pedagogic practice i.e. indicative of a transmissionist versus student-centred or connectionist orientation. The probability of a person succeeding on a particular item is modelled as being dependent only on the difference between the person's ability and the item's difficulty. The mathematical function that relates these two is the logit function or log of the odds. For the simplest Rasch model, involving only items with dichotomous responses, this relationship is expressed as:

$$\ln\left(\frac{P_{ni}}{1-P_{ni}}\right) = B_n - D_i$$

where P_{ni} is the probability of success for person n on item i, B_n is the ability measure of person n and D_i is the difficulty measure of item i. In this study, the Rasch rating scale model (Andrich, 1999; Bond & Fox, 2007; Wright & Mok, 2000) was used, which extends the dichotomous model for use with items with Likert-scale response formats, such as those employed in Pampaka et al's (2012) items. For example, items with a five-point response scale, such as Pampaka et al's, are modelled as having four thresholds. Each item threshold k has its own difficulty estimate F_k and this estimate is modelled as the threshold at which the person has a 50/50 chance of choosing one category over the category below (Bond & Fox, 2007). In the rating scale model, the relative difficulty of each threshold is estimated only once across the entire set of items in the rating scale i.e. the relative difficulty of each threshold is assumed to be the same across all items. The model equation for the rating scale model is therefore expressed as:

$$ln\left(\frac{P_{nik}}{1-P_{nik}}\right) = B_n - D_i - F_k$$

where P_{nik} is the probability of person n choosing a given category on item i, B_n is the ability measure of person n, D_i is the difficulty measure of item i and F_k is the difficulty for threshold k (Bond & Fox, 2007).

The Rasch model is based on three key assumptions of unidimensionality, invariance and local independence (Panayides et al., 2010). The assumption of unidimensionality means that items should all be measuring essentially the same thing. The model requires that items should also discriminate appropriately i.e. that test-items distinguish between persons because having a higher 'ability' entails a higher probability of success (Coe, 2008). Local independence means that items should not provide hints, clues, insights or guidance for the solution of other items (Panayides et al., 2010). For persons, the assumption of invariance implies that item parameters should ideally remain constant across different samples of the relevant population i.e. that, for persons, their relative probabilities of success must be in line with those of others in the population (Coe, 2008).

Fit statistics provide a diagnostic tool for judging how well the data fit the Rasch model. Two types of fit statistics, infit and outfit, are both calculated based on the mean of the squared residuals, but employ slightly different techniques for judging how well a particular item (or person) fit the model. The infit statistic is weighted to give more consideration to the residuals of persons whose ability is near the item's difficulty (Bond & Fox, 2007). The outfit statistic is simply the mean square of the residuals, divided by degrees of freedom (Coe, 2008), and is not weighted. Hence the outfit statistic may be unduly sensitive to extreme outliers and, as a result, the infit statistic is routinely paid more attention (Bond & Fox, 2007). Infit and outfit values are both expected to be close to 1, thus mis-fitting items (or persons) are usually defined as those lying outside an specified interval of values close to 1. In particular, values above 1 indicate a poor fit to the model, whilst those below 1 indicate a better than expected fit or overfit. Following Pampaka et al (2012), in this study, values of infit and outfit higher than 1.3 are taken to indicate misfit. However, what constitutes an acceptable range of values for fit statistics varies in the literature (for example, Bond and Fox, 2007; Smith et al., 1998; Wu and Adams, 2007). Partly for this reason, in this study, misfit is taken as an indicator for further investigation rather than an absolute cut-off point indicating an item should be rejected. In addition, items are usually included to represent various aspects of the construct to be measured. Thus careful consideration is required before throwing them out, since this has implications for content validity and the theoretical conceptualisation of the construct (Bohlig et al., 1998; Pampaka et al., 2012). The approach taken towards the interpretation of fit statistics in this study is discussed further in Chapter 4 alongside the presentation of the results of the Rasch analysis of data from section C of the survey instrument, comprising Pampaka et al's (2012) items.

A major strength of Rasch analysis is that, precisely because the model is underpinned by the assumptions of unidimensionality, invariance and local independence, the veracity of these assumptions may be tested empirically (Panayides et al., 2010). For example, in line with Swan's (2006) original *practices scale*, Pampaka et al (2012) assumed that the opposite end of the spectrum from being transmission-oriented was having a connectionist orientation or being student-centred more generally. Hence, for the purposes of analysis, items with stems describing student-centred or connectionist practices were reverse-coded. However, Pampaka et al (2012) noted their results might indicate a second dimension of connectionism, orthogonal to student-centrism, concluding that the multidimensionality of the scale is worthy of further study. A principal components analysis of the Rasch residuals[3] provides a means for testing to what extent the data conforms to the assumption of unidimensionality and, in particular, for exploring its potential multi-dimensionality.

A second example is the use of Differential Test Functioning (DTF) and Differential Item Functioning (DIF) to test empirically the assumption of invariance across different samples. The items used in this study, conducted with secondary school mathematics teachers, were originally designed by Swan (2006) for use with GCSE mathematics teachers working in Further Education colleges. Pampaka et al (2012) adjusted and validated these items for use with teachers of post-compulsory mathematics. To some extent, the samples in these three studies could be construed as being drawn from different teacher populations. Thus Rasch analysis provides a means of testing to what extent a measure of transmissionist pedagogic practice generalizes across these teacher populations.

Other statistical analyses

Data that could be analysed statistically were manually entered into PASW Statistics 18.0 initially. This package was used to generate descriptive statistics (i.e. frequency distributions and means) and calculate inferential statistics (t-tests and χ^2 tests) where appropriate. Descriptive statistics relating to teachers' ICT use, in terms of their

[3] The difference between a person's observed response to an item and the response predicted by the Rasch model (Bond & Fox, 2007; Coe, 2008).

frequency of ICT use, their orientation towards ICT and their pedagogic practices involving ICT use are reported in Bretscher (2014). An independent data coding check, based on a 10% sample of questionnaires, gave a coding accuracy of greater than 99.9%.

The purpose of constructing the measure of transmissionist self-reported pedagogic practice was to explore associations between a connectionist orientation towards teaching mathematics and teachers' ICT use, in terms of their frequency of ICT use, their orientation towards ICT and their pedagogic practices involving ICT. Once the measure of transmissionist self-reported pedagogic practice was constructed, the individual teacher's measures were imported back into the PASW Statistics 18.0 software. Independent samples t-tests were then carried out comparing the transmissionist measure of frequent and occasional users of ICT; of teachers with a positive and negative orientation towards ICT and of teachers reporting frequent and occasional occurrence of pedagogic practices using ICT. Levene's test for equality of variances was checked. On the occasions where equality of variances could not be assumed the appropriate degrees of freedom and adjusted t-statistics were reported. Similarly, independent samples t-tests and chi-squared tests were used, where appropriate, to explore associations between the transmissionist measure and ICT use with background variables such as gender, age and length of service in the teaching profession. Bonferroni adjustments were not applied, since the intention was to indicate where associations might lie rather than to be conclusive.

3.3.2 Analysis of GeoGebra interviews, lesson observations and post-observation interviews

Data from the GeoGebra interviews were the main focus for qualitative analysis to address RQs 2, 2a and 2b since these interviews provided a common situation across which the case study teachers' use of technology for teaching mathematics could be contrasted. Initially both the GeoGebra interviews and post-observation interviews were transcribed (see Appendices G, I, J for transcription protocol and interview excerpts). A *visual transcript*, a pictorial list of configurations of the angle at the centre theorem made visible by the case study teacher, was also made for the discussion of Diagram 1 in the GeoGebra interview (see Appendix H and Chapter 6 for a more detailed discussion of this process). However, coordinating the visual, on-screen video of the case study teachers' manipulation of the GeoGebra file with the interview transcript for coding purposes proved highly problematic. Thus a narrative of the GeoGebra interview was written as a means of coordinating the visual data with the interview transcript and highlighting key moments in the interview to break down the real-time flow of the video. Writing the narrative meant viewing the video at different speeds i.e. by breaking it down into different grain-sizes of interval. For example, watching the discussion of the first GeoGebra diagram all the way through, without stopping, gave a sense of key moments and the general flow of the interview. It was then possible to zoom in, watching short sequences of the video in order to write the narrative, paying closer attention to key moments, and at times watching the video stop/start to coordinate better the case study teacher's manipulation of GeoGebra with the interview transcript. Zooming out again to watch longer intervals provided a means of checking whether the narrative gave a valid portrayal of the key moments and general flow of the interview. Brief quotes from the interview transcript were included in the narrative as a means of linking what the teachers did – their theories-in-action – with what they said – their espoused theories. For example, configurations of the angle in the centre theorem mentioned in the narrative were those that were both elicited through dragging and identified verbally by the case study teacher (see also Chapter 6).

The narratives of the GeoGebra interviews were coded using the TPACK framework and Knowledge Quartet (see Appendices D, E and F for coding exemplars), focussing primarily on the discussion of Diagram 1, since this diagram was most familiar to the case study teachers. The TPACK framework was chosen to address RQ2 because it provided a means of analysing teachers' knowledge with a particular lens for focussing on technology that other frameworks for teacher knowledge in mathematics education lack (see Chapter 2). The Knowledge Quartet was chosen to complement the TPACK framework, providing a means of producing a fine-grained analysis of teacher knowledge, focusing on their mathematical knowledge in particular to address RQ2a (see also Chapters 1 and 6 for a more detailed discussion supporting this choice). From the lesson observations and post-observation interviews, episodes were identified that were significant in the sense that they could be construed to be informed by a case study teacher's mathematical knowledge for teaching using technology (Rowland et al., 2005). These episodes were then similarly coded using the TPACK framework and Knowledge Quartet. The lesson observations and post-observation interviews were also used to triangulate (Lincoln & Guba, 1985) the finding (see Chapter 4) that the ICT pedagogic practice items in the survey, intended to be teacher-centred, actually appear to reflect dominant or prevalent practices involving ICT. The triangulation involved categorising the case study teacher's use of software and hardware in each lesson observation and their rationale for doing so, as expressed in the post-observation interview, using the ICT pedagogic practice item stems. To address RQ2b, a framework for identifying instances of distributed cognition was developed by mapping the elements of Hutchins' (1995) view of distributed cognition onto the discussion of Diagram 1 in the GeoGebra interviews. The development of this framework, including methodological issues involved in the analysis, is described in more detail in Chapter 7. The data from the GeoGebra interviews then provided a basis for conducting a series of thought experiments to populate this framework and hence to identify minimum knowledge requirements for each of the categories in the framework.

3.4 Summary

This chapter has set out and justified the adoption of a mixed-methods approach towards investigating individual teachers' knowledge and how it is involved in interacting with technology to produce the mathematical knowledge made available in the classroom. It was argued that adopting a mixed methods approach represented a pragmatic methodological means of addressing the research questions identified and justified in Chapters 1 and 2. The research design was justified with respect to the epistemological stance and socio-cultural perspective in relation to technology use outlined in Chapter 1. The methods for data collection and analysis were also described and justified.

Chapter 4 - Exploring associations between ICT use and transmissionist versus connectionist orientations

This chapter uses survey data to explore associations between ICT use and a transmissionist versus connectionist orientation towards teaching mathematics to address Research Question 1:

RQ1 *How is a connectionist orientation towards teaching mathematics associated with teachers' frequency of use of ICT, their orientation towards ICT and their pedagogic practices involving ICT?*

Specifically, this chapter reports the results of constructing a scale of transmissionist self-reported pedagogic practice using Rasch analysis on data from applying Pampaka et al's (2012) items to secondary school mathematics teachers. This measure is then used to explore associations between a transmissionist versus connectionist orientation towards teaching mathematics and teachers' ICT use, in terms of their frequency of ICT use, their orientation towards ICT and their pedagogic practices involving ICT. Case study data from lesson observations and post-observation interviews are used to explore these findings further and to triangulate (Lincoln & Guba, 1985) the findings relating to pedagogic practices involving ICT.

In Chapter 1, it was noted that while Askew et al (1997) defined connectionist and transmissionist orientations mainly in terms of beliefs, in the terms of this study such beliefs are considered as part of teacher 'knowledge'. For example, a connectionist orientation means knowing that understanding mathematics means having a connected knowledge of the subject. Transmissionist teachers provide a contrast with connectionist teachers in this respect, since their knowledge of what it means to understand mathematics is that it concerns the acquisition of a collection of routines or procedures.

A connectionist orientation towards teaching mathematics has not been investigated in relation to ICT use. This is surprising given the prominence of ICT and links made between ICT and student-centred practices (Becker et al., 1999; Law et al., 2008). In Chapter 1, it was argued that there is an imperative for exploring a connectionist orientation in relation to technology. Teachers do integrate some types of digital technology into their classroom practice. However, these types of digital technology are commonly assumed to be teacher-centred (rather than student-centred) and to maintain or even encourage existing 'traditional' pedagogies. Remillard's (2005) perspective suggests that connectionist-oriented teachers may work to shape digital technologies commonly assumed to be teacher-centred, using them in ways that conform to their own more student-centred pedagogy. If this is the case, then the apparent deficit in teachers' use of digital technologies in terms of the extent to which their potential is realised may be over-estimated. In addition, one might expect connectionist teachers to make more frequent use of mathematical analysis software (Pierce & Stacey, 2010), such as spreadsheets, dynamic geometry software and graphing software.

4.1 The development of a measure of transmissionist self-reported pedagogic practice

This section briefly summarises the research leading to the development and validation of Pampaka et al's (2012) set of items. In this study, these items were used to construct a scale of transmissionist self-reported pedagogic practice for secondary school mathematics teachers, using Rasch analysis. The results of the measure construction are reported in Section 4.2.

Combining the work of Ernest (1991) and Askew et al (1997), Swan (2006) operationalised connectionist, transmissionist and discovery orientations in the form of a questionnaire on teacher beliefs, as well as developing and validating a set of self-report items on pedagogic practices. Swan used the questionnaires as a means for tracking the changes wrought by a professional development intervention with a group of 64 GCSE mathematics teachers working in Further Education colleges in England. The self-report items on pedagogic practices were conceptualised as representing either teacher-centred classroom behaviours, arising from transmission-oriented beliefs, or student-centred classroom behaviours, arising from a constructivist position. Swan constructed a *practices scale* from these items and related teachers' scores on this scale to their categorisation, via the beliefs questionnaire, as transmission, discovery or connection-oriented. For his sample of FE teachers, he found that, as expected, transmission-oriented teachers reported a greater frequency of teacher-centred practices, but that although discovery teachers reported an increase in student-centred practices, connectionist teachers were the most student-centred of all. Thus on Swan's scale of teacher versus student-centred practice, transmissionist and connectionist teachers appear on opposite ends of the scale whilst discovery teachers occupy an indeterminate, 'middle' position, not related to distinctive practices.

Building on Swan's work, Pampaka et al (2012) revised and validated the set of self-report items on pedagogic practice for measuring teachers' practices in post-compulsory, 'advanced' level mathematics college classrooms. Using data from 110 responses to this revised set of items, they employed a Rasch measurement approach to create an interval measure of transmissionist self-reported pedagogic practice. Whilst Pampaka et al's data showed an acceptable fit to the assumption of uni-dimensionality implemented in Rasch analysis, they interpret some of their results as potential evidence of a second dimension of connectionist practice that might be orthogonal to student-centrism, concluding that the multidimensionality of the scale is worthy of further study. Pampaka et al (2012, p. 484) state they "therefore proceed with caution, bracketing for the moment whether the opposite end of the spectrum from 'transmissionism' [in their study] is 'connectionism' or 'student-centrism' more generally. However, given Swan's (2006) findings in relation to his practices scale, for the purposes of this study, it is reasonable to assume that the opposite end of the spectrum from a transmission-orientation is a connectionist orientation. Finally, in Chapter 3, it was noted that Pampaka et al's (2012) items relate to teachers' classroom practices rather than to the knowledge that underpins a connectionist or transmissionist orientation. Nevertheless, the teachers' responses to these items provide an indicator of their espoused theories (Argyris & Schon, 1974; Pampaka et al., 2012) of teaching practice, hence they may be taken as an indirect indication of their mathematical knowledge for teaching.

4.2 Results of measure construction: a scale of transmissionist self-reported pedagogic practice

This section reports the results of constructing a scale of transmissionist self-reported pedagogic practice using Rasch analysis on data from applying Pampaka et al's (2012) items to secondary school mathematics teachers. In particular, through a discussion of fit statistics and other means available through Rasch analysis (see also Chapter 3), it is argued that Pampaka et al's (2012) items could constitute a reasonable scale for measuring a construct which, following their lead, is called 'transmissionist self-reported practice in secondary mathematics teachers'.

The Rasch analysis of Pampaka et al's (2012) set of pedagogic practice items achieved a person reliability score of 0.83 suggesting an acceptable overall level of consistency and reliability. Table 4.1 shows the item measures, fit statistics and point-measure correlation resulting from the Rasch analysis of the data. The fit statistics for all except six items were below the 1.3 threshold of concern. Similarly, for all except two items, the point-measure correlation was above the 0.2 threshold. The misfitting items are discussed in the following paragraph. Taken together, these results suggest that Pampaka et al's items could constitute a reasonable scale for measuring the construct of 'transmissionist self-reported practice in secondary mathematics teachers'.

The six items identified as mis-fitting according to the 1.3 threshold are shown in bold in Table 4.1 (C6, C10, C22, C23, C24, C26). The item stems are shown in Figure 4.1 and Appendix A. Of these items, two in particular, C6 'I encourage students to work more slowly' and C24 'I cover only the important ideas in a topic', seem like possible candidates for deletion due to their point-measure correlation being well below 0.2 and indeed close to zero. However, deleting these two items had no practically significant effect on the summary statistics of the Rasch analysis. Re-running the Rasch analysis without C6 and C24 marginally improved the person reliability to 0.84; however the item separation decreased from 7.70 to 7.55. Similarly, there were no practically significant changes to the measures and fit of the other items.

Items C6, C22, C24 and C26 were also mis-fitting in Pampaka et al's data. They argued on theoretical and methodological grounds that these items should not be excluded at this point, since they may belong to a secondary dimension of connectionist teaching, yet may also be interpreted differently by some discovery or even transmission-oriented teachers (Pampaka et al, 2012). For example, they suggest that C6 'working more slowly' and C24 'covering only the important ideas' may be seen as part of a laissez-faire, discovery approach rather than encouraging more thoughtful work. In addition, a transmission-oriented secondary maths teacher preparing pupils for high-stakes examinations at KS4 might construe C24 positively as revising 'only the important ideas' for the exam. Pampaka et al argue item C22 'I find out which parts students already understand and don't teach those parts' indicates an approach to formative assessment as an important part of connectionist teaching, which may not be present in all student-centred teaching since this also incorporates discovery-oriented approaches.

Table 4.1 Item measures, fit statistics and point-measure correlation

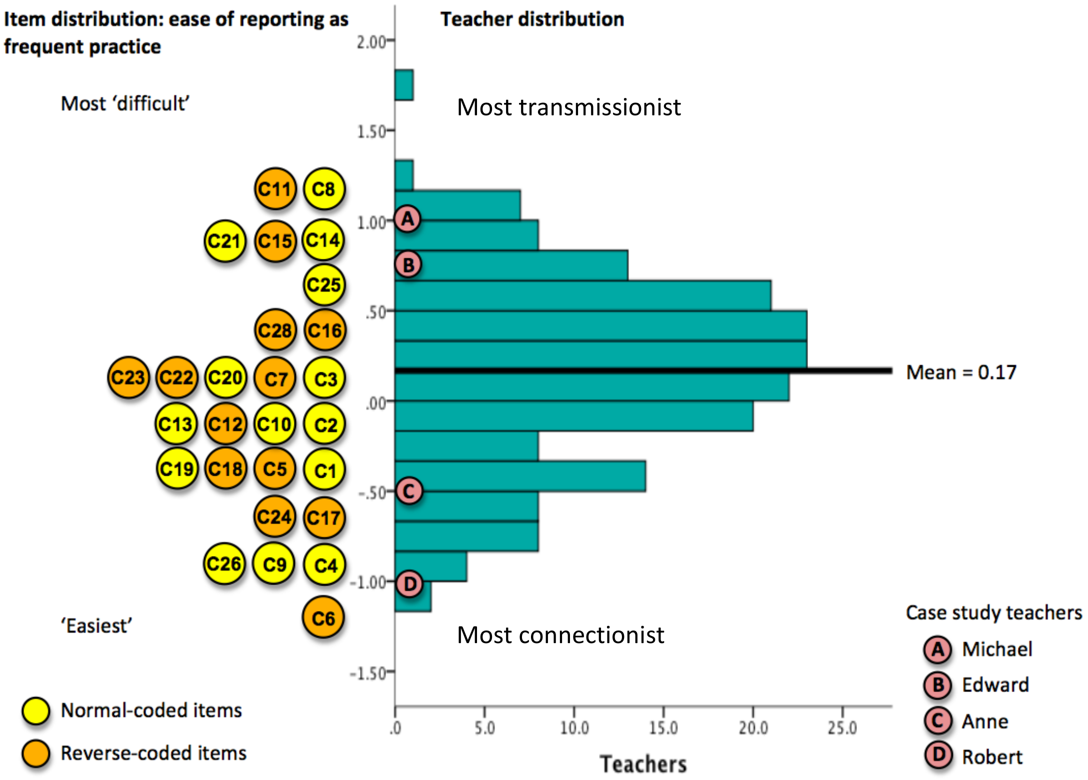
Item	Raw score	Count	Measure	Model S.E.	Infit MNSQ	Outfit MNSQ	Pt-Measure Correlation
C1	647	180	-.43	.09	.97	.98	.54
C2	616	181	-.20	.08	.87	.86	.41
C3	552	181	.23	.08	.77	.77	.53
C4	719	183	-.92	.09	.93	.90	.40
C5	656	183	-.42	.08	.94	.96	.55
C6	**738**	**183**	**-1.09**	**.10**	**1.39**	**1.42**	**.06**
C7	567	183	.16	.08	.68	.70	.55
C8	401	182	1.17	.08	1.14	1.10	.44
C9	720	183	-.93	.09	.74	.74	.43
C10	**624**	**182**	**-.22**	**.08**	**1.31**	**1.37**	**.29**
C11	403	181	1.13	.08	.99	.96	.38
C12	621	183	-.18	.08	.75	.72	.51
C13	627	183	-.22	.08	1.02	1.02	.49
C14	442	183	.92	.08	.94	.93	.67
C15	448	183	.88	.08	.70	.69	.56
C16	520	183	.45	.08	.73	.74	.50
C17	696	182	-.76	.09	.83	.82	.53
C18	632	181	-.31	.08	.78	.81	.49
C19	647	183	-.36	.08	.85	.87	.66
C20	546	183	.29	.08	1.00	1.03	.30
C21	447	181	.85	.08	.90	.90	.50
C22	**545**	**182**	**.28**	**.08**	**1.39**	**1.40**	**.29**
C23	**560**	**181**	**.16**	**.08**	**1.47**	**1.49**	**.24**
C24	**683**	**180**	**-.72**	**.09**	**1.44**	**1.49**	**.02**
C25	483	180	.62	.08	.93	.92	.56
C26	**700**	**182**	**-.79**	**.09**	**1.52**	**1.50**	**.34**
C28	526	183	.41	.08	1.11	1.12	.32
Mean			.00	.08	1.00	1.01	
S.D.			.65	.01	.25	.26	

The same argument applies to item C23 ‘I teach each student differently according to individual needs’, suggesting a tailored approach to teaching, in line with formative assessment, not necessarily present in discovery-oriented approaches. Item C26 (‘knowing exactly what maths the lesson will contain’) is intended as a transmissionist item, suggesting the lesson is controlled to exclude non-standard mathematics. However, Pampaka et al suggest some student-centred teachers may instead interpret this item as regarding subject matter knowledge, i.e., that they should have knowledge of all the mathematics that ‘might’ arise in the lesson. Similarly, although intended as a transmissionist item, some student-centred teachers may also interpret C10 ‘I try to cover everything in a topic’ as a need to provide sufficient coverage of an examination syllabus or scheme of work.

Rasch analysis provides a means of exploring the possibility of a secondary dimension of connectionist teaching via a principal components analysis (PCA) of the Rasch residuals. The PCA of Rasch residuals is a means of extracting the common factor explaining the most residual variance, under the assumption there is a meaningful structure in the residuals that could constitute another dimension. If this factor is found merely to ‘explain’ random noise, then there is no meaningful structure in the residuals (Linacre, 1998). Unlike a standard PCA on raw data, the absolute size with which an item loads on the factor is inconsequential for a PCA of Rasch residuals. Instead, it is the patterning of the loadings that is important, in particular, contrasting positive and negative loadings (Linacre, 2011). For the data-set in this study, the PCA of Rasch residuals produced a factor or ‘first contrast’ with an eigenvalue of 3.4, slightly larger than 2, the rule of thumb used as the smallest amount that could be considered to indicate a dimension (Raiche, 2005; Linacre, 2011). Transmissionist items had mainly positive loadings and connectionist items had mainly negative loadings, suggesting there might be some meaningful structure indicative of a possible second dimension in the data. However, this evidence was inconclusive on the presence of a second dimension since, for example, it was not clear whether the patterning might simply be an artefact of the reverse-coding of connectionist items (see Figure 4.1 for reverse coded items).

In conclusion, a decision was made to retain the six mis-fitting items in the model at this point. Although there is some statistical evidence to suggest their exclusion, in agreement with Pampaka et al’s argument, this was outweighed on the grounds of theoretical and methodological considerations. In particular, the six items may still contribute to aspects of a transmissionist scale of pedagogic practice, thus they are retained for the purpose of maintaining content validity (Bohlig et al., 1998). Though a second dimension in the data might exist, the evidence remains inconclusive and requires further research. In addition, retaining the complete set of items is advantageous in maintaining the possibility of comparison across data sets for the purposes of future research.

Figure 4.1 shows the distribution of both items and secondary mathematics teachers on the resulting measurement scale of transmissionist self-reported pedagogic practice, in a diagram adapted from the item-person map provided by the *Winsteps* software. On the right hand side, the distribution of teachers is displayed as a histogram, with higher positioning on the scale indicating more transmissionist practice. Conversely, lower positioning on the scale is indicative of more connectionist practice. On the left hand side, the approximate position of items is shown, with positioning on the scale indicating the ease of reporting frequent occurrence in practice. In other words, items placed low on the scale were relatively easy to report as frequently occurring, whilst those placed high on the scale were relatively difficult to report as frequently occurring. For reverse-coded items, the opposite holds e.g. C15. Case study teachers are also represented on the scale as red points. The positive mean person measure (0.17), displayed on the histogram, indicates that the set of test-items was slightly too ‘easy’ for the target sample. That is, the most transmissionist teachers have too few ‘difficult’ items to differentiate them; whilst there are too few sufficiently connectionist teachers to provide good information about the ‘easiest’ items. Another interpretation of the mean person measure is that the population of teachers is somewhat skewed towards transmissionist pedagogic practices. Nevertheless, the test seems reasonably well-targeted at the sample. Differential Test Functioning, involving the comparison of item measures across samples, suggests that the measure of transmissionist pedagogic practice did not vary significantly between Pampaka et al’s (2012) sample of post-compulsory mathematics teachers and the sample of secondary mathematics teachers used in this study (see scatter graph comparing item measures in Appendix K).



C1	Students work through exercises.		C15	Students discuss their ideas.
C2	Students work on their own, consulting a neighbour from time to time.		C16	Students work collaboratively in pairs.
C3	Students use only the methods I teach them.		C17	Students invent their own methods.
C4	Students start with easy items and work up to harder questions.		C18	Students work on substantial tasks that can be worked on at different levels.
C5	Students choose which questions they tackle.		C19	I tell students which questions to tackle.
C6	I encourage students to work more slowly.		C20	I encourage students to work more quickly.
C7	Students compare different methods for doing questions.		C21	I go through only one method for doing each question.
C8	I teach each topic from the beginning, assuming they know nothing.		C22	I find out which parts students already understand and don't teach those parts.
C9	I teach the whole class at once.		C23	I teach each student differently according to individual needs.
C10	I try to cover everything in a topic.		C24	I cover only the important ideas in a topic.
C11	I draw links between topics and move back and forth between topics.		C25	I teach each topic separately.
C12	Students work collaboratively in small groups.		C26	I know exactly what maths the lesson will contain.
C13	I avoid students making mistakes by explaining things carefully first.		C28	I jump between topics as the need arises.
C14	I tend to follow the textbook closely.			

Note: C27 was not included in this study

Figure 4.1 Scale of transmissionist self-reported practice for secondary maths teachers

4.3 Results

The purpose of constructing the measure of transmissionist self-reported pedagogic practice was to explore associations between teachers' ICT use and their orientation towards teaching mathematics. Before presenting the results of this, it is worth giving a brief indication of how the transmissionist measure and ICT use are associated with background variables such as gender, age and length of service in the teaching profession. Further statistical information is provided in Appendix M.

Female teachers were significantly more connectionist than their male colleagues (df = 180, t = -2.87, p = .005). Significantly more female teachers reported low levels of confidence than men in using ICT in both lessons with an IWB (n = 181, df = 3, χ^2 = 10.1, p = .018) and lessons in a computer suite (n = 174, df = 4, χ^2 = 10.9, p = .028). For lessons in a computer suite, significantly more female teachers than men agreed with the statement that *students' lack of familiarity with software makes lessons difficult* (n = 174, df = 4, χ^2 = 13.0, p = .011). There were no clear differences between male and female teachers in the frequency of their use of hardware and software or their reporting of pedagogical practices involving ICT.

There was no significant association between age or length of service and the transmissionist measure. More teachers than expected with less than six years experience report using ICT more frequently than their colleagues, significant at the 5% level (n = 183, df = 3, χ^2 = 10.6, p = .014). This greater frequency of ICT use by less experienced teachers appears to translate into more frequent use of IWBs (n = 187, df = 3, χ^2 = 10.9, p = .012) and use of IWB software (n = 183, df = 4, χ^2 = 12.3, p = .015) - in a whole-class context with an IWB as opposed to in a computer suite. In both cases, more teachers than expected with less than six years experience use IWBs and IWB software in almost every lesson. In a similar vein, significantly more teachers than expected aged less than 30 used PowerPoint frequently in a computer suite (n = 174, df = 4, χ^2 = 11.6, p = .021) and, approaching significance at the 5% level, with an IWB (n = 182, df = 4, χ^2 = 9.44, p = .051). There is a belief that a change in generation may be necessary for ICT to become more fully integrated into classroom practice (e.g. Sancho, 2010). If so, these results suggest that this belief may be overly optimistic. Whilst a new generation of secondary mathematics teachers in this survey might use ICT more frequently, it does not appear to be in ways that mathematics education research suggests would be most desirable.

4.3.1 A surprising association between frequent use of teacher-centred software and a more connectionist orientation.

A general association seems to hold across almost all types of hardware and software that teachers who make frequent use of ICT tend to be more connectionist than occasional users. Tables 4.2, 4.3 and 4.4 show the results of t-tests comparing the mean transmissionist measure of frequent and occasional users of software when used in conjunction with an IWB, of software used in a computer suite and of hardware respectively. For hardware and software used on an IWB, frequent use corresponds to the concatenation of categories 'once per week' and 'almost every lesson', with occasional use corresponding to categories 'never', 'annually' and 'once or twice per term'. For two exceptions, graphic calculators and Logo with an IWB, due to very low levels of use, 'once or twice per term' was included as indicating frequent use, so that occasional use corresponds to categories 'never' and 'annually' only. Similarly for software use in computer suites, due to levels of use being lower overall, frequent use was considered to include 'once or twice per term'.

Although the difference in mean transmissionist measure between frequent and occasional users of ICT does not always reach statistical significance, in almost all cases of hardware and software the difference is negative, indicating frequent users have a lower mean measure and thus a more connectionist orientation than occasional users. The exception to this pattern was laptops (see Table 4.4) where frequent users had a slightly higher mean transmissionist measure than occasional users, however this result was not statistically significant and frequent users were only 13 in number. In addition, comparing themselves to their departmental colleagues, teachers who regarded themselves as more frequent users of ICT ($n = 61$) tended to be more connectionist than those who did not ($n = 118$), although this was not a significant difference at the 5% level ($df = 177$, $t = -1.87$, $p = .064$).

Table 4.2 Comparing mean transmissionist measure of frequent and occasional users of software when used in conjunction with an IWB

IWB: frequency of software use	**n freq, occ**	**Mean difference freq - occ**	**t-stat**	**df**	**p-value**
CD-Roms	36; 139	-.100	-1.01	173	.313
Database	23; 151	-.088	-.738	172	.462
Email	53; 119	-.086	-.991	170	.383
Graphing software	49; 122	-.017	-.185	169	.853
Dynamic geometry	30; 145	-.324	-3.13	173	.002*
IWB software	146; 33	-.285	-2.86	177	.005*
Logo	15; 151	-.196	-1.36	164	.175
MyMaths	116; 64	-.284	-3.58	178	p<.001*
Other websites	112; 61	-.315	-3.92	171	p<.001*
PowerPoint	107; 72	-.251	-3.21	177	.002*
SMILE	11; 152	-.130	-.770	161	.443
Spreadsheet	45; 134	-.174	-1.92	177	.056
Word	74; 105	-.119	-1.50	177	.137

* indicates statistical significance at the 5% level. Occasional user = (never, annually, once or twice per term); Frequent user = (once per week, almost every lesson). **Except Logo**, where Occasional user = (never, annually); Frequent user = (once or twice per term, once per week, almost every lesson)

Surprisingly, the association between frequent use of ICT and a more connectionist orientation *is* statistically significant for what is generally assumed to be teacher-centred software. Specifically, frequent use of IWB software, PowerPoint, the *MyMaths* website and ‘Other websites’ in a whole-class context with an IWB showed a statistically significant association with more connectionist self-reported pedagogic practice – see Table 4.2. Similarly for lessons in a computer suite, teachers who reported frequent use of the *MyMaths* website, Word and ‘Other websites’ were significantly more connectionist than occasional users – see Table 4.3 – with IWB software also approaching the 5% significance level. It is surprising that such software appears to be associated with a more connectionist orientation since it is usually assumed to maintain and even encourage existing transmission-oriented pedagogies.

In terms of mathematical analysis software, frequent use of dynamic geometry software was significantly associated with a more connectionist orientation, both in a whole class context with an IWB and giving students direct access to software in a computer suite. In lessons with an IWB, spreadsheets approached statistical significance at the 5% level, with frequent use again associated with a more connectionist orientation than occasional use, although use of graphing software was not significantly associated with the transmissionist measure – see Table 4.2. In a computer suite, frequent use of both spreadsheets and graphing software were significantly associated with a more connectionist orientation. In addition, Email, SMILE and Logo also showed statistically significant associations between frequent use and a more connectionist orientation, however teachers making frequent use of the latter two types of software were in a very small minority – see Table 4.3. These results are not unexpected since maths education research suggests that such software is compatible with and may even support more connectionist practices.

Table 4.3 Comparing mean transmissionist measure of frequent and occasional users of software when used in a computer suite

Computer suite: frequency of software use	**n freq, occ**	**Mean difference freq - occ**	**t-stat**	**df**	**p-value**
CD-Roms	33; 136	-.129	-1.26	167	.208
Database	29; 143	-.013	-.124	170	.902
Email	45; 123	-.231	-2.54	166	.012*
Graphing software	88; 85	-.240	-3.02	171	.003*
Dynamic geometry	83; 90	-.323	-4.20	171	p<.001*
IWB software	69; 102	-.158	-1.94	169	.054
Logo	19; 152	-.330	-2.60	169	.010*
MyMaths	129; 44	-.233	-2.54	171	.012*
Other websites	126; 48	-.328	-3.78	172	p<.001*
PowerPoint	90; 82	-.123	-1.51	155.5	.133
SMILE	15; 152	-.461	-3.24	165	.001*
Spreadsheet	103; 72	-.227	-2.83	173	.005*
Word	89; 84	-.216	-2.72	171	.007*

* indicates statistical significance at the 5% level. Occasional user = (never, annually); Frequent user = (once or twice per term, once per week, almost every lesson)

Interpreting these results, it is important to bear in mind the levels of usage of the different types of software. In lessons with an IWB, for teacher-centred software such as IWB software, the *MyMaths* website, PowerPoint and ‘Other websites’, the majority of teachers were frequent users, with occasional users in the minority, and indeed these software were the most frequently used in this context. In comparison, most teachers were only occasional users of mathematical analysis software with an IWB, a finding in common with other surveys reporting low integration of such software in classroom practice (Miller & Glover, 2006). Similarly, for lessons where pupils are given direct access to software in a computer suite, the *MyMaths* website and ‘Other websites’ were the most frequently used software, again with mathematical analysis software reported as having lower levels of usage.

Table 4.4 Comparing mean transmissionist measure of frequent and occasional users of hardware

Frequency of hardware use	**n** **freq, occ**	**Mean difference** **freq - occ**	**t-stat**	**df**	**p-value**
IWB	158; 25	-.300	-2.71	181	.007*
Data projector	67; 114	-.089	-1.10	179	.271
Computer suite (shared)	32; 150	-.199	-1.98	180	.050
Computer suite (maths only)	10; 170	-.172	-1.02	178	.311
Laptops	13; 170	.049	.328	181	.743
Graphic calculator	32; 149	-.267	-2.66	179	.009*

* indicates statistical significance at the 5% level. Occasional user = (never, annually, once or twice per term); Frequent user = (once per week, almost every lesson). **Except graphic calculators**, where Occasional user = (never, annually); Frequent user = (once or twice per term, once per week, almost every lesson)

In terms of hardware, teachers who report making frequent use of IWBs were significantly more connectionist than occasional users, see Table 4.4. Interpreting this statistic, it should be remembered that frequent users of IWBs formed the vast majority: 158 teachers compared to 25 occasional users. Of the 25 occasional users, 15 were clustered in two schools. The teachers in one of these schools had access to IWBs; in the other school, they had access to data projectors but not to IWBs. The remaining 10 occasional users were among the respondents from 8 different schools. The mean difference in transmissionist measure between frequent and occasional users of shared computer suites approached statistical significance at the 5% level, again with frequent users reporting more connectionist pedagogic practices. However, the skew in usage of shared computer suites is in the opposite direction to IWBs with occasional users forming the vast majority. Frequent use of graphic calculators also appears to be significantly associated with a more connectionist orientation, however again some care is needed in interpreting this result. Usage of graphic calculators was so low that a 'frequent' user corresponds with anything more than annual usage. Detailed descriptive statistics on both the frequency of hardware and software use are reported in Bretscher (2014).

4.3.2 Associations between teachers' orientation towards ICT and the transmissionist measure.

A general association also appears to hold between having a positive orientation towards ICT and having a more connectionist orientation towards teaching mathematics. Tables 4.5 and 4.6 display the results of t-tests comparing the mean transmissionist measure of teachers reporting a positive orientation and those reporting a negative orientation towards ICT in the context of teaching mathematics in a whole-class context with an IWB and in the context of giving students direct access to software in a computer suite respectively. A positive orientation towards ICT corresponds to the concatenation of categories 'strongly agree' and 'agree'; whilst a negative orientation corresponds to categories 'strongly disagree', 'disagree' and 'neither agree nor disagree'. For items worded in the negative sense, indicated in Tables 4.5 and 4.6 by italics, a negative orientation towards ICT corresponds to the concatenation of categories 'strongly agree', 'agree' and 'neither agree nor disagree'; whilst a positive orientation corresponds to categories 'strongly disagree' and 'disagree'.

For almost all items regarding ICT orientation, teachers responding positively were somewhat more connectionist-oriented than those responding negatively, although again this general association was not always statistically significant. The negative difference in mean measure between these two groups indicates that teachers reporting a positive orientation towards ICT have a lower mean measure and thus a more connectionist orientation than those reporting a negative orientation. The two exceptions to this pattern were both negatively worded items. Firstly, teachers who showed a negative

ICT orientation, by agreeing with the statement *Students' lack of familiarity with software makes lessons involving ICT difficult*, were slightly more connectionist than those who disagreed with the statement, though not significantly. In the following section, allowing students to take control over the IWB more frequently is seen to be significantly associated with a more connectionist orientation, thus it is possible that connectionist-oriented teachers face more difficulties in the classroom if students lack familiarity with the software. The second exception was those who disagreed that ICT lessons in a computer suite take more time to prepare, interpreted as a positive ICT orientation, were slightly more transmissionist than those who expressed agreement with the statement, although again this was not significant.

Table 4.5 Comparing mean transmissionist measure of positive and negative orientation towards ICT when using an IWB in lessons

IWB: ICT orientation	**n** **pos, neg**	**Mean difference** **pos - neg**	**t-stat**	**df**	**p-value**
confident using ICT	152; 24	-.061	-.551	174	.583
more time needed for preparation	52; 126	-.082	-.971	176	.333
ICT contributes to learning	135; 45	-.232	-2.63	178	.009*
ICT improves engagement	147; 34	-.169	-1.72	179	.087
students' lack of familiarity with software	80; 101	.013	.162	179	.872
ICT helps understanding	140; 40	-.230	-2.51	178	.013*
classroom management more difficult	149; 31	-.278	-2.76	178	.006*
cover more ground	99; 81	-.110	-.143	178	.153

* indicates statistical significance at the 5% level. Negative ICT orientation = (SD, D, N); Positive ICT orientation = (A, SA). *For negatively worded items in italics* Negative ICT orientation = (SA, A, N); Positive ICT orientation = (D, SD)

Both in the context of using ICT with an IWB and in a computer suite, those who responded positively that ICT makes an important contribution to students' learning and helps them understand mathematics were significantly more connectionist than those who responded negatively. These results are not unexpected since maths education research suggests that when ICT is used in ways that generally accord with connectionist practices, it may support students' understanding of mathematics. In addition, teachers who disagreed that classroom management was more difficult in a whole-class context with an IWB (i.e. were positively inclined towards ICT use) were significantly more connectionist than those who either agreed or who expressed no preference either way.

Table 4.6 Comparing mean transmissionist measure of positive and negative orientation towards ICT when for lessons in a computer suite

Computer suite: **ICT orientation**	**n** **pos, neg**	**Mean difference** **pos - neg**	**t-stat**	**df**	**p-value**
confident using ICT	140; 32	-.253	-2.47	170	.015*
more time needed for preparation	51; 122	.079	.892	171	.374
ICT contributes to learning	129; 44	-.219	-2.38	171	.018*
ICT improves engagement	135; 38	-.127	-1.31	171	.192
students' lack of familiarity with software	73; 99	-.045	-.551	170	.582
ICT helps understanding	127; 45	-.196	-2.15	170	.033*
classroom management more difficult	81; 92	-.120	-1.49	171	.139
cover more ground	36; 135	-.232	-2.36	169	.020*

* indicates statistical significance at the 5% level. Negative ICT orientation = (SD, D, N); Positive ICT orientation = (A, SA). *For negatively worded items in italics* Negative ICT orientation = (SA, A, N); Positive ICT orientation = (D, SD)

Bretscher (2014) found that teachers were significantly less confident, found classroom management more difficult and felt they covered less ground (i.e. curriculum material) when using ICT in a computer suite compared to using ICT in a whole-class context with an IWB. The results in Table 6 show that for ICT lessons in a computer suite, teachers who expressed confidence in using ICT and who agreed that they cover more ground were significantly more connectionist than those who did not. However, the perception of difficulties with classroom management in a computer suite showed no association with the transmissionist measure. These results suggest having a more connectionist orientation might enable teachers to overcome barriers of confidence and perceptions of covering less ground in a computer suite compared to in a whole-class context with an IWB. However, having a connectionist orientation may not support teachers in overcoming perceived difficulties with classroom management when giving students direct access to ICT in a computer suite compared with using ICT in a whole class context with an IWB.

4.3.3 Dominant as opposed to teacher-centred practices: associations between ICT pedagogic practices and the transmissionist measure.

Originally the ICT pedagogic practices items were intended to be either teacher or student-centred, in a similar manner to Swan's (2006) items, which Pampaka et al (2012) revised to construct their measure of transmissionist self-reported pedagogic practice. Tables 4.7 and 4.8 display the results of t-tests comparing the mean transmissionist measure of teachers reporting frequent occurrence of ICT pedagogic practices with those reporting occasional occurrence in their ICT lessons with an IWB and in a computer suite respectively. Frequent occurrence corresponds to categories 'almost always' and 'most of the time'; whilst occasional occurrence corresponds to categories 'almost never', 'occasionally' and 'half the time'.

Of those items that showed a statistically significant association with the transmissionist measure, all were intended as student-centred items, with frequent occurrence of these practices associated with having a more connectionist orientation. None of the ICT pedagogic practice items, in particular not one of the items intended to be teacher-centred, showed a statistically significant association between frequent occurrence and having a more transmissionist orientation. More specifically, for lessons with an IWB, two of the items originally intended to be teacher-centred *I use ICT for presentation purposes* and *I control the software on the IWB* were the most frequently occurring practices across all teachers. Similarly, for lessons in a computer suite, the items *Students' use ICT to practise skills* and *I provide precise instructions for software use* were the most frequently occurring and were originally intended to be 'teacher-centred'. None of these four items showed a statistically significant association with the transmissionist measure. This suggests that rather than being teacher-centred per se, these items may reflect dominant or prevalent practices involving ICT, that occur frequently in any teachers' classroom practice irrespective of their connectionist or transmissionist orientation.

Table 4.7 Comparing mean transmissionist measure of teachers reporting frequent and occasional occurrence of pedagogic practices using ICT with an IWB

IWB practices	n freq, occ	Mean difference freq - occ	t-stat	df	p-value
teacher presentation	138; 40	.038	.406	176	.685
student discussion	78; 102	-.312	-4.15	178	p<.001*
teacher control	146; 33	.060	.491	40.3	.626
explore students' ideas	50; 130	-.250	-2.94	178	.004*
prevent discrepancies	74; 96	-.013	-.166	168	.868
students control	39; 141	-.338	-3.71	178	p<.001*
highlight discrepancies	55; 118	-.152	-1.79	171	.075
avoid mistakes	77; 99	.135	1.70	174	.090

* indicates statistical significance at the 5% level. Occasional = (almost never, occasionally, half the time); Frequent = (most of the time, almost always)

Table 4.8 Comparing mean transmissionist measure of teachers reporting frequent and occasional occurrence of pedagogic practices using ICT in a computer suite

Computer suite practices	n freq, occ	Mean difference freq - occ	t-stat	df	p-value
practise skills	92; 75	-.028	-.337	165	.737
work collaboratively	90; 82	-.236	-3.04	170	.003*
'get a feel' for the software	80; 92	-.258	-3.34	170	.001*
explore discrepancies	22; 143	-.271	-2.31	163	.022*
individual work	70; 100	-.063	-.782	168	.436
investigate problems	61; 111	-.212	-2.60	170	.010*
provide precise instructions	90; 77	.093	1.15	165	.252
avoid technical difficulties	51; 118	-.014	-.164	167	.870

* indicates statistical significance at the 5% level. Occasional = (almost never, occasionally, half the time); Frequent = (most of the time, almost always)

The next section uses case study data from lesson observations and post-observation interviews to triangulate (Lincoln & Guba, 1985) the finding that the ICT pedagogic practices items intended to be teacher-centred actually appear to reflect dominant or prevalent practices involving ICT, that occur frequently in any teachers' classroom practice irrespective of their connectionist or transmissionist orientation. The triangulation involved categorising the case study teacher's use of software and hardware in each lesson observation and their rationale for doing so, as expressed in the post-observation interview, using the ICT pedagogic practice item stems. The final section of this chapter explores these dominant practices in relation to mathematical knowledge for teaching using technology, finding that even these practices, whether they employ 'teacher-centred' software or mathematical analysis software (Pierce & Stacey, 2010), appear to involve significant knowledge on the part of the teacher.

The four case study teachers (Robert, Anne, Edward and Michael) were chosen along two dimensions of variation that might be associated with mathematical knowledge for teaching using technology. Thus the case study teachers were chosen to be two of the most connectionist-oriented – Robert and Anne - and two of the most transmissionist-oriented teachers – Edward and Michael - of those who volunteered (see Figure 4.1). The level of school support provided the other dimension of variation (see Table 3.3, Chapter 3).

4.4 Dominant practices: using ICT to enhance pedagogic aspirations

In common with Askew et al's (1997) original study, no straightforward distinction could be made between the pedagogic practices involving ICT of the two connectionist case study teachers, Robert and Anne, and the two transmissionist teachers, Edward and Michael. At times, during their IWB and computer suite lessons, each of the case study teachers employed dominant pedagogic practices in line with the survey findings described in the previous section. In particular, none of the teachers went beyond using ICT to enable general pedagogic aspirations (Ruthven, 2009) towards using ICT to transform their mathematics pedagogy, with one possible exception. Thus, in each of the lessons, ICT was used to enhance pedagogy, making it more effective, more efficient and more appealing, rather than to transform practice.

Robert was one of the most connectionist teachers in the survey sample (see Figure 4.1, transmissionist measure -1.01). In his lesson with an IWB, he used a web-based pupil-response system, which he designed with help from colleagues, to test his Year 12 pupils' knowledge of basic rules of differentiation. The pupil-response system collated the students' answers to multiple choice questions and presented an analysis of the results, for the class as a whole or for an individual student, which Robert could display on the IWB if he chose. He then used a PowerPoint presentation, recapping the basic rules of differentiation and introducing the procedure for carrying out the Chain Rule for differentiation as the main point of the lesson.

Similarly, Michael, one of the most transmissionist teachers in the sample (see Figure 4.1, transmissionist measure +1.01), used an ActivInspire presentation to recap coordinates and introduce translations as a type of geometric transformation to his Year 7 class. He also used a pupil-response system, commercially produced by Activote, to test whether his pupils could read coordinate points from a set of axes and identify translation vectors for various pairs of congruent triangles positioned in the first quadrant.

In both cases, the teachers used *ICT for presentation purposes* and maintained overall *control of the software* in accordance with the survey finding of dominant practices. The PowerPoint and ActivInspire software aided their demonstrations, making them more effective through use of colour and accurate diagrams and more efficient since the teacher saved time in the lesson by not having to write out or draw the procedures by hand. For example, Robert used colour in his PowerPoint presentation to draw students' attention to patterns in notation in the basic rules for differentiation and the Chain Rule procedure. Michael used facilities for creating accurate diagrams and dynamic possibilities to demonstrate triangles moving under translation to create a more effective presentation. Although Michael asked pupils to the board to translate the triangles, he ended up directing their movement of the shape so that in effect this was a teacher demonstration. Both teachers used the pupil-response systems to engage pupils in question-and-answer sessions that practised the demonstrated procedures and to assess pupils' progress in a manner that was more effective and efficient than was possible using other means. In this sense, both teachers used ICT to enhance their pedagogic practices, which were, at least superficially, very similar in these lessons.

Edward was one of the more transmission-oriented teachers in the survey sample (see Figure 4.1, transmissionist measure 0.74). His mathematics department was not equipped with IWBs, having data projectors installed instead, which Edward in any case preferred. In his lesson with a data projector, he chose to use a pre-prepared GeoGebra file (see Figure 4.2), downloaded from the MEI website, to convince his Year 12 students empirically that the graph of an inverse function is a reflection in the line $y = x$ of the original function. Here, Edward also used *ICT for presentation purposes*, maintaining *control of the software*, in accordance with the survey finding of dominant practices. He controlled the software, using GeoGebra to graph a function $y = f(x)$. He brought students to the board, intending them to sketch by hand the graph of $y = f^{-1}(x)$ onto the axes projected on the normal whiteboard once they had found the inverse function by algebraic means. Then using GeoGebra, Edward dragged a point on the graph of $y = f(x)$, causing a reflection of this point in the line $y = x$ to trace out the reflection of the function. He drew their attention to the fact that the reflection of the graph $y = f(x)$ and their sketch graph of the inverse function matched. In the lesson, this worked as intended for Edward's first example of a straight-line graph. In following examples, the students instead attempted to sketch the reflection in the line $y = x$ of $y = f(x)$ on the board, only subsequently finding the equation of the inverse function algebraically. This meant that using the trace simply verified the correctness of their sketch of the reflection rather than providing the empirical evidence Edward's rationale for using the GeoGebra file required. Nevertheless, his intention was to enhance his teacher presentation of the mathematical topic, making it more effective by *accentuating features*, providing vivid images and striking effects to highlight properties and relations (Ruthven & Hennessy, 2002).

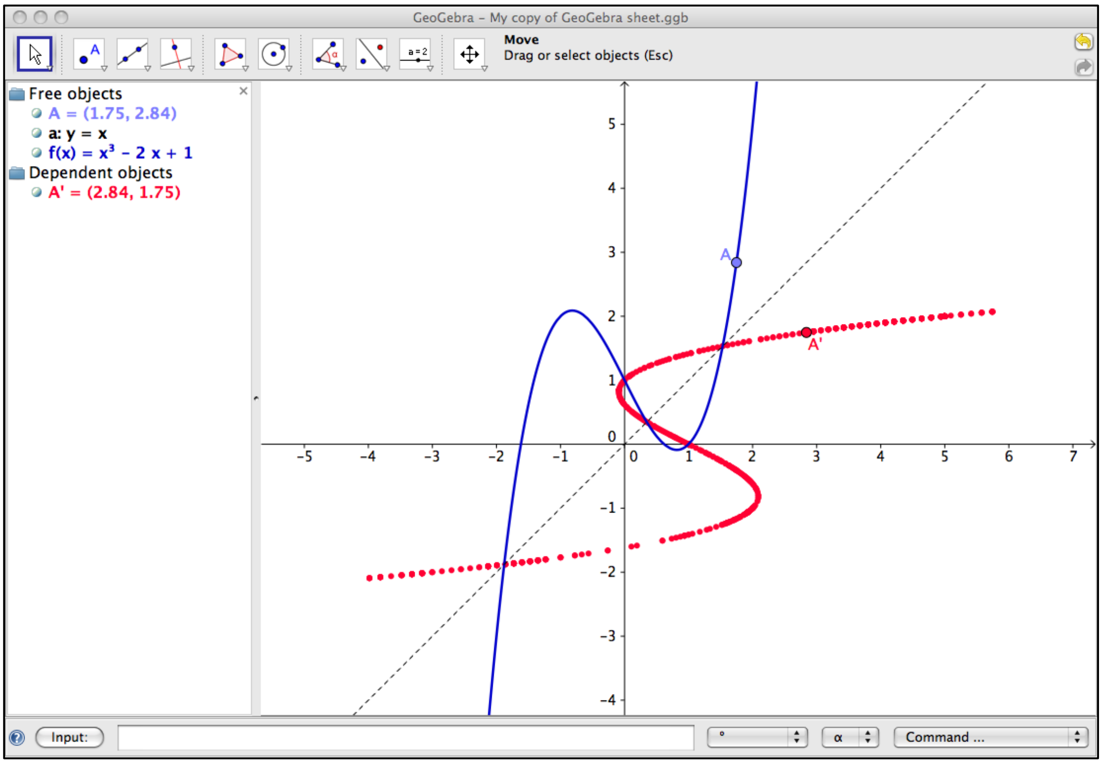


Figure 4.2 Edward's GeoGebra file on graphs of inverse functions

Anne, one of the more connectionist-oriented teachers in the sample (see Figure 4.1 transmissionist measure -0.50), did not do a lesson centred on using software with an IWB in a whole-class context in the sense that the other teachers did. In her two computer suite lessons, however, she also used PowerPoint slides and teacher presentation on an IWB to give instructions for software use amongst other things. This coincides with one of the dominant practices from the survey, *I provide precise instructions for software use*, involving ICT use in a computer suite. Across her two lessons in the computer suite, Anne intended her students to use Autograph as a means of empirically establishing the rule that $y = f(x) + a$ is a transformation of $y = f(x)$ through a translation of $+a$ units vertically and 0 units horizontally, by experimenting first with straight line graphs of the form $y = mx + c$ and then with quadratics of the form $y = x^2 + c$. Anne aimed to use the software to enhance her general pedagogic aspirations, by establishing the rule that $y = f(x) + a$ is a transformation of $y = f(x)$ through a translation of $+a$ units vertically more efficiently and effectively. Anne hoped that by removing the need to graph the functions by hand, her students would spot the rule more quickly and, similar to Edward, that using the software would accentuate the rule, for example, by juxtaposing the visual representations of functions with their equations, thus establishing it more effectively. During the course of the two lessons, Anne had specified that pupils should input equations as '*y* =' into Autograph. The worksheet she gave to pupils introduced function notation $y = f(x)$ and $y = f(x) + 2$ alongside straight line equations $y = x$ and $y = x + 2$ as a means of leading pupils towards a more general definition of the translation rule. However, many of her pupils interpreted the function notation as a cue to enter $y = f(x)$ and $y = f(x) + 2$ into Autograph, without first defining $f(x)$. Unexpectedly, instead of rejecting the undefined function, the Autograph software responded by plotting horizontal lines $y = 1$ and $y = 3$, apparently setting $f(x) = 1$ by default. This resulted in confusing and counterproductive output on the students' screens and their sketches on the worksheet. Seeking to avoid further disruption to the lesson, Anne issued more precise instructions to only enter equations as '$y =$' and not to enter function notation into Autograph.

In their computer suite lessons Robert, Michael and Edward all used software for the purpose of *students using ICT to practise skills* in accordance with the survey finding of dominant practices. The planned activities were similar in that the teachers hoped that use of the software would enhance general pedagogic aspirations in terms of *intensifying engagement*, *effecting activity* and *establishing ideas* (Ruthven, 2009).

Michael took his Year 8 students into the computer room so that they could interact with an online lesson on volume of cuboids provided by the *MyMaths* website. His pupils were to work through the explanatory slides included as part of the lesson, before practising calculating the volumes of cuboids on subsequent worksheet-style slides and other skill-practice games. Michael adopted an almost discovery-oriented approach to the *MyMaths* lesson, hoping that interacting with the software would help the students be more autonomous in their learning. In particular, he liked a dynamic diagram embedded in one of the slides, which led towards developing the formula for the volume of a cuboid by demonstrating volume as building up layers of unit height.

Edward took his Year 9 students to revise straight-line graphs using a game named *Save our Dumb Planet* provided by the *MangaHigh* website, fronted by a team including Marcus du Sautoy (see Figure 4.3). In the early stages, the game provided an impetus for pupils to practise substituting co-ordinates into equations of the form $y = mx + c$, verifying the correct equation passing through two points to determine the path of a meteor destined to collide with Earth. In addition to practising problems of this type, Edward hoped his pupils might indirectly gain experience of how the coefficients m and c effect the graph of the equation $y = mx + c$.

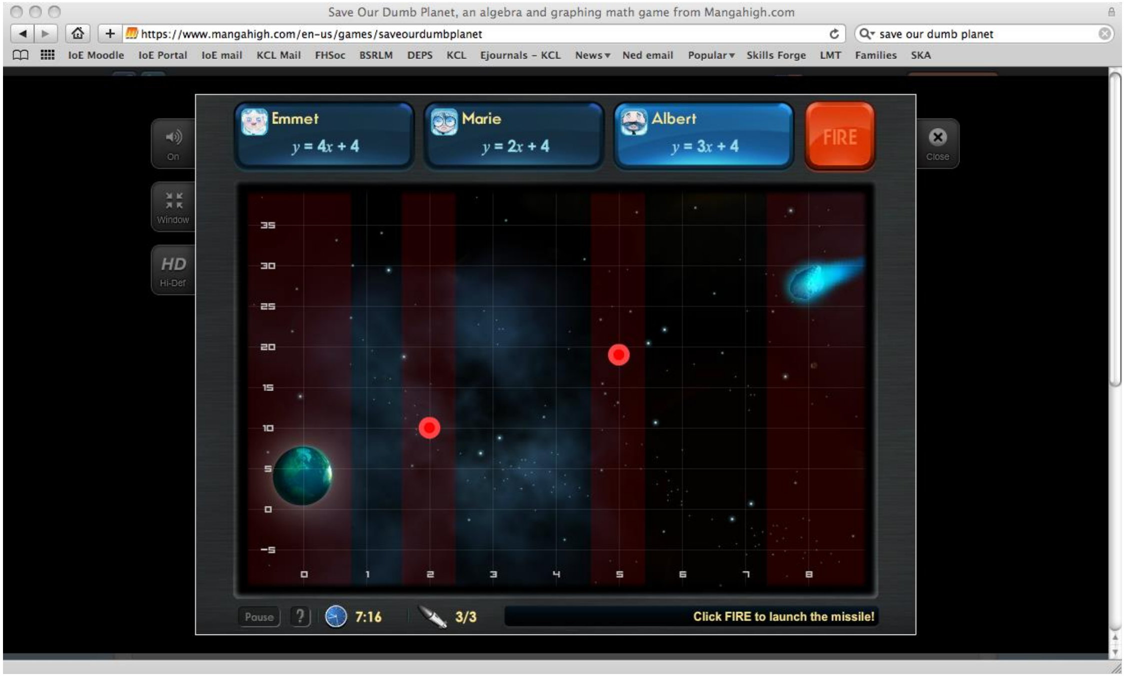


Figure 4.3 Save our Dumb Planet from Edward's computer suite lesson

Robert's computer suite lesson was with a Year 8 group learning about reflection. For the second part of his lesson, Robert intended his pupils to practise constructing the reflections of various shapes (e.g. triangles) using GeoGebra files that he had prepared in advance. He described these files as being similar to a paper-and-pencil exercise but with the advantage that pupils were able to use the software to check and correct their work. With more difficult tasks where the line of reflection was diagonal, he had added line segments connecting corresponding points on the object and image of reflection. Robert's aim was that these line segments would help pupils to construct the reflections correctly and in addition reinforce an understanding the line of reflection as the perpendicular bisector of the line segments. During the lesson, technical difficulties resulted in the pupils' computer screens freezing so that they were unable to engage with this task.

4.4.1 An exception to dominant practice: Robert's use of GeoGebra

For the first part of his computer suite lesson, Robert had created a series of maze activities, embedded in GeoGebra files, designed to take advantage of students' intuitive, tacit understandings of reflection as a means of making these understandings explicit and thus leading towards a more formal understanding of reflection. Using the mouse to direct the movement of a point, coloured in blue, the pupils had to guide the blue point's reflection, shown in red, successfully through a maze (see Figure 4.4.). The reflection line was super-imposed on the maze diagram and the path of the red point was traced. Robert hoped that the activity would encourage pupils to *predict* how the reflected red point would move in relation to movement of the blue point as a means of increasing their chances of completing the maze successfully. By predicting the movement of the red and blue points, he hoped his pupils intuitive understandings of reflection would be made more explicit.

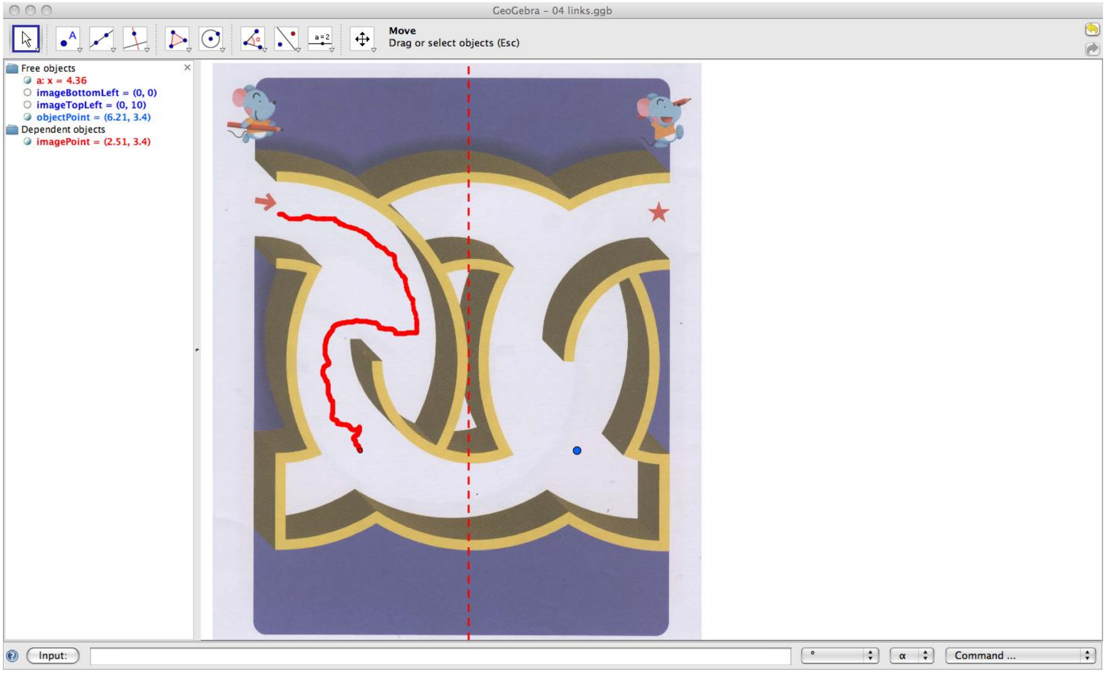


Figure 4.4 One of Robert's GeoGebra maze activities - by dragging the blue dot, guide the reflected red dot through the maze

The mazes were intended to progress in difficulty, with later files incorporating a diagonal line of reflection and a line segment joining the pair of reflected points. Robert hoped that using the line segment would aid students in their negotiation of the maze and so prompt their recognition of the line of reflection as the perpendicular bisector of the red and blue points. Robert’s use of GeoGebra appears to be an exception because he uses the mathematical affordances of the software to affect his pupils’ learning in a way that would not be easy to achieve without digital technology, in comparison to the lessons described above where software is used to replicate and enhance paper-and-pencil activities.

4.5 Indications of mathematical knowledge for teaching using technology

The case study lessons described in the previous section provide indications that even the dominant pedagogic practices involving ICT reported in the survey, that do not go beyond using ICT to enhance general pedagogic aspirations, require significant knowledge on the part of the teacher. This assertion is illustrated below using the case study lessons as telling cases. The nature and content of this knowledge is explored in more detail in the next chapter using data from the semi-structured interviews on using GeoGebra files to teach circle theorems.

On the surface, Robert and Michael's pedagogic practices involving ICT appeared very similar in their IWB lessons. Looking in more detail, however, they are quite different in terms of the mathematical knowledge for teaching using technology in evidence during the lesson. Robert decided that purchasing a pupil-response system of the kind Michael had access to would be too costly and not of sufficient benefit for his school. Instead, he thought smart phones would be a cheaper and more mathematically versatile alternative to the voting handsets Michael handed out to his pupils, in particular since Robert observed that most of his pupils already owned such devices. Along with a colleague, Robert designed and programmed a web-based interface where he could set questions to which his pupils could respond via their smart phones and which collated their responses, presenting the data in ways that could be accessed and interpreted during a lesson. Robert thought that requiring his students to enter mathematical notation using a smart phone would result in errors that would detract from his pedagogic purpose of enhancing learning through formative assessment. For this reason, he opted for multiple-choice questions, avoiding this difficulty whilst still allowing him to pose complex questions by carefully choosing common errors as "distractors" (Rob-IWB-int, 13.6.2012) to inform him of where his students might be going wrong. During his IWB lesson, Robert displayed a bar chart of the results for at least one question, noting the modal answer was correct and going through the distractors in turn. Thus as a designer, Robert had given careful consideration to how the mathematical format of the questions, given the technology available, could best meet his pedagogical purposes.

For Michael, the lesson observation provided an impetus to use the Activote pupil-response system available at his school. He thought using the system would engage his pupils and provide some useful assessment data. Although coordinates and translation vectors seemed like a good topic to assess using multiple-choice questions, Michael had found it difficult to create standard notation column vectors when devising a quiz within the restrictions of the Activote software. In the end, he resorted to coordinate-style notation both for coordinates and translation vectors, introducing a potential confusion. Creating the quiz, entering pupil data for identifying their answers and setting up the handsets for the lesson took substantially more effort and time than a typical lesson. Michael had not given much consideration to the design of the multiple-choice questions, generating distractors almost at random by swapping coordinates and the positive/negative sign. Thus he was unable to use the assessment information in the lesson beyond noting what percentage of the class had chosen the correct answer.

As described in the previous section, in the second of Anne's two computer suite lessons, she encountered some difficulties with Autograph when pupils entered $y = f(x)$ and $y = f(x) + 2$ without defining $f(x)$ and unexpectedly the software responded by plotting horizontal lines $y = 1$ and $y = 3$, apparently setting $f(x) = 1$ by default instead of rejecting the undefined function. In this contingency, Anne's reaction was to cut off any further confusion by simply telling her pupils it was wrong to enter $y = f(x)$. This seemed a reasonable reaction at the time, but in the post-observation interview Anne was dissatisfied with her response. Anticipating and preparing for this complexity, would have required some knowledge of the rules governing the software's interpretation of mathematical notation and its relevance to the teaching of the lesson, for example, in terms of providing more precise instructions for software use or perhaps delaying the introduction of function notation or indeed using the software's recognition of function notation as a means to explore transformations of functions.

Another instance of mathematical knowledge for teaching using technology arose in Anne's first lesson in the computer suite. Using software to graph functions tends to focus attention on the global shape of the graph, possibly obscuring a local understanding of the graph as a set of coordinate points. This has particular relevance to the topic of transformations of graphs, since a connection is being made between applying a transformation locally to a set of graphical coordinates and the function in terms of its equation. Anne wanted her pupils to investigate transformations of graphs by experimenting first with straight-line graphs of the form $y = mx + c$, so that she could make a connection with their previous work on the effect of the coefficients m and c. At the end of the lesson, she asked her pupils what transformation would bring $y = x$ to $y = x + 2$ (see Figure 4.5). Encouraged by using the software, her pupils attended to the global shape of the graph, suggesting various possibilities for the transformation including a vertical translation of +2 units, a horizontal translation of -2 units or a reflection in the line $y = x + 1$. Whilst each of these transformations does map the graph $y = x$ onto $y = x + 2$, considered locally as a transformation of coordinates, only a vertical translation of +2 units can be connected easily to the equation $y = x + 2$ in a manner that extends to other, non-linear functions.

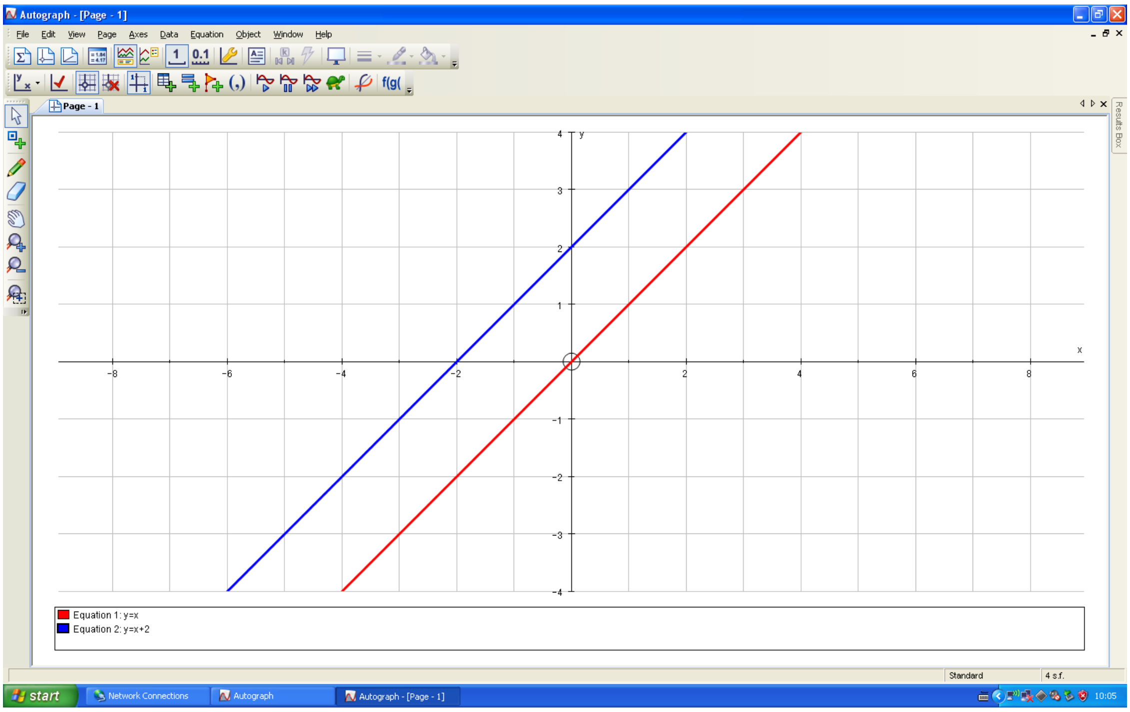


Figure 4.5 Autograph output in Anne's CS lesson

For example, writing $y = f(x)$ where $f(x) = x$, it is possible to consider $y = x + 2$ as $y = f(x + 2) + 0$, i.e. a horizontal translation of $y = x$ by -2 units, however this is relatively unhelpful in linking the transformation to its equation in the form $y = mx + c$ and does not extend to other functions such as $y = x^2 + c$. Similarly, in terms of the global shape of the graph, reflecting $y = x$ in the line $y = x + 1$ appears to be a reasonable answer, resulting in the line $y = x + 2$. However, considered locally as a transformation of coordinates, it is difficult to use function notation to write down a reflection in the line $y = x + 1$ in a way that can be meaningfully connected to the equation $y = x + 2$ and certainly not in a way that extends to other functions.

Anne acknowledged these options by recording them on the IWB, however she appeared to steer her students towards a vertical translation of +2 units, sidelining the other responses without explanation. From her lesson plan, Anne did not appear to have anticipated these responses. In the post-observation interview, Anne appeared to be planning to resolve the question in the next lesson by extending the investigation to quadratics of the form $y = x^2 + c$, thereby justifying the choice of a vertical translation in the case of straight-line graphs as the only option capable of generalising and hence the most useful option mathematically. In the second lesson, whilst some of the pupils did go on to investigate quadratics of the form $y = x^2 + c$, Anne again steered her students towards the vertical translation through whole-class discussion and the structuring of her worksheets, thus a principled argument for choosing the vertical translation remained elusive. Anticipating the pupils' responses and having a pedagogic strategy for countering their conjectures in a principled way in these lessons, would seem to require some knowledge of how pupils' attention is drawn to the local and global features of functions when presented graphically using the Autograph software.

Similarly, in Edward's IWB lesson with GeoGebra on inverse functions, attending to the global shape of the graph, pupils quickly spotted from the first example that the graph of $y = f^{-1}(x)$ was likely to be the reflection of $y = f(x)$ in the line $y = x$. Justifying this conjecture requires attending to the graph locally as a set of coordinate points, an argument that was not made explicit during the lesson.

Finally, in his computer suite lesson, Edward hoped his pupils might be reminded indirectly of how the coefficients m and c effect the graph of the equation $y = mx + c$ by repetitively substituting coordinates into straight-line equations and watching the graphs being plotted as they played the skill-practice game *Save our Dumb Planet*. In the post-observation interview, he thought that whilst they would be unlikely to gain experience of gradient as "rise over run" or "1 unit along and m units up" (Ed-CS-int, 27.6.2012), they might be reminded of the difference between positive and negative gradients at least. Although axes were provided for the graphical representation of the meteor's path, they did not make it easy to read off coordinates and on closer inspection the axes were not square, making it difficult to interpret both local and global features of the straight-line graph. Knowledge of how the software generated examples of coordinates and straight-line graphs, e.g. randomly or by design, at least to the extent that there were questions including straight-line graphs with both positive and negative gradients; regarding the quality of the graphical representation in the game; and of how this related to his pedagogic purposes might reasonably be assumed to have informed Edward's post-lesson judgement.

4.6 Summary

The survey findings suggest that frequent use of hardware and software has a general association with a more connectionist orientation. Surprisingly, frequent use of software commonly assumed to maintain and even encourage existing transmission-oriented pedagogies is also associated with a more connectionist orientation. Whilst there are pedagogic practices involving ICT associated with a connectionist orientation, items on pedagogic practices involving ICT that were designed to be ‘teacher-centred’ appear to have no association either way. These ‘teacher-centred’ practices involving ICT may instead be construed as ‘dominant’ practices, in that they are also the most frequently occurring across all teachers.

Case study data supports the survey finding of dominant practices, suggesting that, superficially at least, there is little difference between connectionist and transmission teachers use of ICT. Both connectionist and transmission-oriented teachers used software in IWB and computer suite lessons to enhance general pedagogic aspirations rather than to transform mathematics pedagogy, with one possible exception. The case study lessons also provide indications that even these dominant practices, whether they involve ‘teacher-centred’ software or mathematical analysis software (Pierce & Stacey, 2010) place significant demands on teacher knowledge.

Chapter 5 - The nature and content of mathematical knowledge for teaching using technology

This chapter begins to address Research Question 2 by investigating the nature and content of individual teachers' *own* knowledge in relation to using technology to teach mathematics.

RQ2 *What is the nature and content of teachers' mathematical knowledge for teaching using technology, as represented by the central TPACK construct?*

In Chapter 1, borrowing from Shulman (1986, p.13), mathematical knowledge for teaching using technology was assumed not only to be a matter of knowing how – being competent in teaching mathematics using technology - but also of knowing what and why. In other words, mathematical knowledge for teaching using technology, as defined in this study, is when know-how or knowledge-in-action is underpinned by and coincides with the teacher's articulated knowledge. The TPACK framework was identified as a suitable means for analysing and comparing teachers' espoused theories (articulated knowledge) and their theories-in-action (knowledge-in-action), in order to make inferences about individual's mathematical knowledge for teaching using technology (see Chapter 1). Finally, contrasting the two transmissionist with the two connectionist case study teachers' use of technology should make visible individual teachers' *mathematical knowledge for teaching using technology*, as argued in Chapter 1, allowing RQ2 to be addressed.

The first section of this chapter analyses case study teacher Robert's computer suite lesson to suggest that a positive stance towards technology, in terms of global aspects of teacher knowledge (e.g. Bowers & Stephens, 2011; Zbiek & Hollebrands, 2008), may not be sufficient to ensure a teacher's use of technology enhances mathematical instruction. In addition, the first section of this chapter suggests that using technology places significant demands on individual teachers' own knowledge for teaching mathematics.

The second section of this chapter goes on to explore the nature and content of mathematical knowledge for teaching using technology, by analysing data from the four case study teachers' semi-structured GeoGebra interviews using the TPACK framework. The analysis focuses on exemplifying the dyadic construct TCK as a means of exploring the nature of the central TPACK construct. This section argues that what distinguishes TPACK from the dyadic construct TCK appears to be mathematical knowledge, abstract in the sense that it generalises across particular technological contexts and mathematical topics. In addition, it serves to highlight the *situated* nature of TPACK, as a transformation (Rowland et al., 2005; Shulman, 1987) of mathematical knowledge for the purposes of teaching using technology. Similar arguments can be made in relation to the other two dyadic constructs PCK and TPK, however they have not been included in this chapter for the sake of brevity.

In the final section of this chapter, the apparent duality in seeing TPACK simultaneously as abstract, mathematical knowledge and yet as *situated* in the context of teaching using technology is addressed using Noss and Hoyles' (1996; & Kent, 2004; & Pozzi, 2002) notion of situated abstraction and Adler's (1999; 2001) concepts of the dilemma of transparency and visibility/invisibility. The criticisms of the weak theorisation of the TPACK framework (Graham, 2011; Ruthven, 2014; Voogt et al., 2012) are acknowledged, however it is argued that this represented a key affordance by allowing the user to move between an integrative and transformative perspective (Graham, 2011) in particular in relation to the central TPACK construct.

5.1 The significance of mathematical knowledge for teaching using technology as represented by the central TPACK construct

The case study lessons, described at the end of Chapter 4, suggest that even carrying out dominant practices, that do not go beyond enhancing general pedagogic aspirations, place significant demands on individual teachers' own knowledge in relation to teaching mathematics using technology. In Chapter 2, the review of research literature identified a group of studies with a common focus on teachers' global conceptions of mathematics as a discipline and on teachers' beliefs about the nature of teaching and learning mathematics with technology (Zbiek & Hollebrands, 2008). In terms of the present study, beliefs and conceptions are considered as part of teacher knowledge. This group of studies (Zbiek & Hollebrands, 2008) serves to highlight the important role teachers' conceptions play in determining the extent and nature of technology integration in classroom practice. However, these studies do not tend to focus on teachers' knowledge of specific mathematical concepts in relation to technology, either in relation to their espoused theories or theories-in-action. Thus, research on technology in mathematics education, whilst exploring the relationship between global aspects of teachers' espoused theories and theories-in-action and their technology use in some depth, has paid relatively little attention to teachers' knowledge of specific mathematical concepts in relation to technology. This is an important omission since the documented shifts in teachers' views suggest a move towards models of teaching aimed at developing conceptual understanding. Such models may require a great deal of knowledge for successful implementation (Thompson & Thompson, 1996) and as noted earlier, inconsistencies between teachers' professed beliefs and practices may be the result of lacking sufficient knowledge necessary to implement them (Thompson, 1992). Whilst highlighting the role of teachers' conceptions in technology integration is important, this section argues that the significance of mathematical knowledge for teaching using technology should not be overlooked nor underestimated.

For example, Bowers and Stephens (2011, p. 290) assert that the central TPACK construct is an empty set with regard to (teachable) knowledge and skills, emphasising instead that teacher educators should seek to nurture a favourable conception of "technology as a critical tool for identifying mathematical relationships". Whilst it may be that teacher educators should seek to nurture favourable conceptions towards using ICT in their trainees, the knowledge required to put such conceptions into practice should not be neglected. Robert's creation and use of the GeoGebra maze activities in his computer suite lesson on reflections demonstrates the kind of favourable conception of technology that Bowers and Stephens (2011) advocate. However, using the maze activities in practice was not trivial and Robert did not entirely succeed in making explicit the mathematical relationships the pupils were exploring using the GeoGebra software. The difficulties Robert experienced provide evidence suggesting mathematical knowledge for teaching using technology plays a significant role in technology integration, perhaps alongside the kind of favourable orientation that Bowers and Stephens (2011) describe.

Robert's series of maze activities, created using GeoGebra, appear as an exception amongst the case study teachers' lessons because he used the mathematical affordances of the software to affect his pupils' learning in a way that would not be easy to achieve without digital technology. In particular, he took advantage of the affordance of the software to preserve geometric relationships under drag to build upon his pupils' intuitive, tacit understandings of reflection as a means of making these understandings explicit and thus leading them towards a more formal understanding of reflection. The aim of his lesson was for pupils to be able to construct the image of a simple shape given the object and reflection line and to recognise the reflection line as the perpendicular bisector of line segments connecting corresponding points on the object and image. A second GeoGebra activity was intended as skill-practice for constructing the image under reflection and to reinforce understanding of the reflection line as a perpendicular bisector. Due to the disruption caused by technical difficulties there was no time for the second activity. This skill-practice activity and the maze activities are described in more detail in Chapter 4.

In the post observation interview, Robert explained what inspired him to create the maze activities. He provided a critique of similar GeoGebra activities lacking an impetus to focus attention on and articulate tacit understandings:

> I had a look on the GeoGebra wiki and most things tended to be 'Here's a mirror line, here's a shape, if you drag this, what's happening?' just kind of … and say what you see. And I could imagine them sitting there with that and basically just dragging the mouse a bit and seeing it happen and ... and then where does it go from there? [Rob-CS-int, 13.6.2012]

He also described a pedagogic strategy of *predict-then-test* that he aimed to use in the lesson to make pupils' understandings of mathematical relationships explicit

> just you know introduce that pause of what do we think is going to happen and then let's test that it's going to happen [Rob-CS-int, 13.6.2012]

and how he intended to formalise these understandings during the lesson by introducing mathematical vocabulary

> So one of the things I wanted to talk about was that if you're moving that point parallel to the mirror line, the point moves in the same direction, whereas as soon as you're moving it in a direction that's not parallel, the point doesn't move in the same way. [Rob-CS-int, 13.6.2012]

Summarising at the end of the lesson, he did introduce mathematical vocabulary during class discussion, in a similar way to the intention described above, describing the movement of the red and blue points. In later more challenging mazes, Robert had included a line segment joining the blue and red points as a potential aid to maze completion (see Figure 5.1). Further, he juxtaposed two identical mazes, one with and one without the line segment, in an attempt to prompt the students to notice the value of the line segment in solving mazes and to question what was special about the line segment that made it helpful. He had hoped to use whole class discussion to lead them towards recognising the line of reflection as the perpendicular bisector of the line segment but ran out of time in part due to the technical difficulties he experienced.

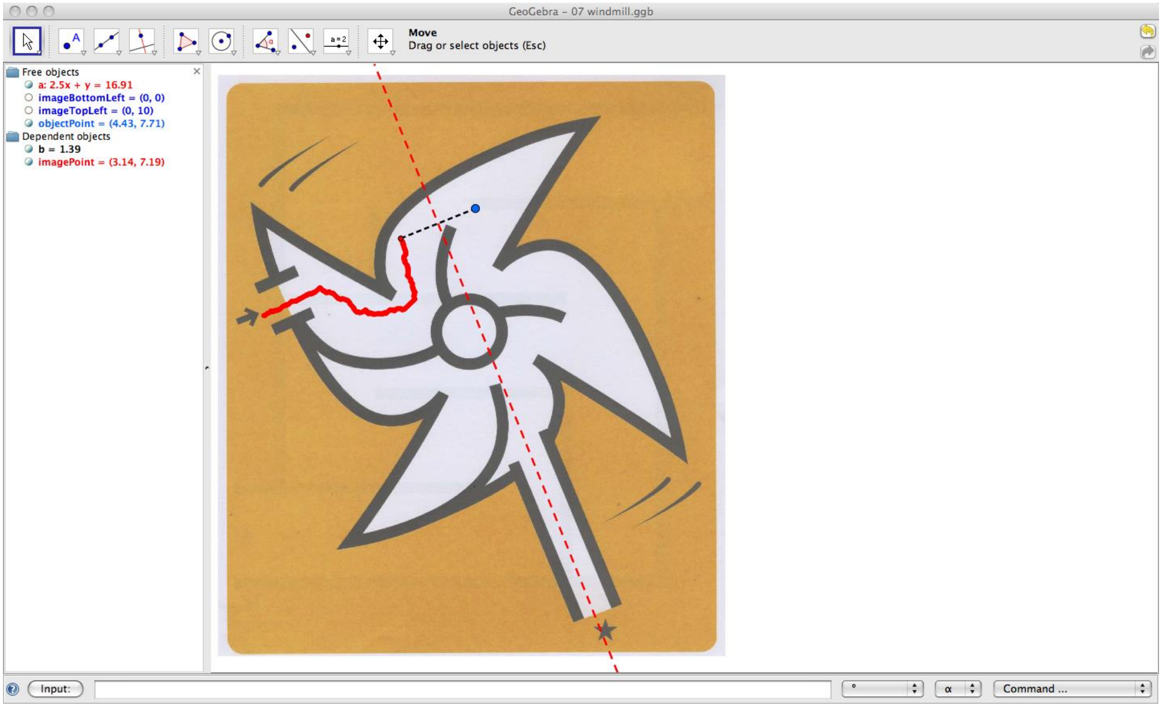


Figure 5.1 Robert's GeoGebra maze with additional line joining red and blue points

During the lesson, Robert did intervene on an individual basis to prompt this realisation:

> R: I tried it with one pupil and it was quite interesting, before I explained what that line was or before we'd had a talk through about what that line was, I don't think he found it helpful at all, but I think, in particular, it was useful, when you were trying to go diagonally through a … so you had to move the point diagonally kind of down through a passageway in the maze, if you just kept … but if you moved the point along that support line, it made it incredibly easy.
>
> I: Right.
>
> R: And so I think once I'd explained what was going on to him, he actually did … and the quality of his path was much better in the second attempt. But I think it's … I don't think any of those pupils made that connection themselves, what that line was, or how it might be helpful. [Rob-CS-int, 13.6.2012]

Thus Robert's design of the maze activities, his use of them in the lesson and his comments about the lesson in the post-observation interview demonstrate the strong emphasis he placed on the use of technology to explore the mathematical relations behind the mathematical phenomenon of reflection, consistent with Bowers and Stephen's (2011) description of a favourable conception of technology. During the post-observation interview, Robert described another GeoGebra activity on the topic of reflection that he uses with higher attaining pupils, designed to provide an impetus for realising the line of reflection as the perpendicular bisector of the line segments connecting corresponding points on the object and image:

> for example, possibly turn it the other way around, right, this is the object, this is the image, where is the mirror line? How can we use the computer to determine where it is? What are the kind of construction methods that you'd use to work out where that mirror line might be? [Rob-CS-int, 13.6.2012]

This suggests that the observation lesson was not an anomaly in Robert's practice but indicative of a coherent conception of "technology as a critical tool for identifying mathematical relationships" (Bowers & Stephens, 2011, p. 290).

Using the series of maze activities successfully to meet the aims of the lesson depended on transforming students' strategies for completing the mazes into more formal understandings of reflection that could be used as strategies for constructing the image given an object and line of reflection. As indicated above in excerpts from the post-observation interview, Robert recognised his interventions with individual pupils and directing whole class discussion as being critical to effecting this transformation.

The maze activities potentially addressed two complementary strategies for using geometric properties to construct the image given the object and line of reflection:

1) using the local geometry of the object together with the properties of reflection, namely, preservation of length and of direction parallel to the line of reflection and reversal of direction in the axis perpendicular to the line of reflection, to construct the image; and

2) using the geometric property that the line of reflection is the perpendicular bisector of line segments connecting corresponding points on the object and image.

The first strategy was addressed through the maze activities by the necessity of considering how to drag the blue point, i.e. in what direction and how far, to guide the reflected red point through the maze. In particular, the main challenge in completing the maze is derived from the reversal of direction caused by the reflection. Less obvious perhaps is that length is preserved: dragging the blue point causes the red point to move the same distance. The second strategy was addressed in later maze activities by the addition of the line segment connecting the blue and red points as a possible aid to maze completion.

Robert was not satisfied with his interventions during the lesson. In the post-observation interview, he pointed to the technical difficulties, his desire to let the students enjoy the maze activities and his rush to move onto the second activity as contributing to the result that he did not spend as much time as intended on discussing the geometric implications of the pupils' maze-solving strategies. Timing was certainly a factor and the technical difficulties meant that he was unable to direct a whole class discussion juxtaposing the identical mazes with and without the line segment joining the red and blue points. As a result, Robert was unable to address the second strategy outlined above involving recognition of the line of reflection as the perpendicular bisector of the line segment joining the red and blue points. However, he did have two opportunities during the lesson to elicit the geometric properties of reflection that underpin the first strategy through whole class discussion.

The first opportunity came when Robert brought the class back together after some time engaging with the maze activities. He displayed one of the early maze activities with a vertical line of reflection and asked pupils to give instructions to a pupil-volunteer to direct their movement of the blue point (see Figure 5.2). Robert summarised their responses, drawing their attention to the relative direction of movement of the red and blue points i.e. that when the blue point was dragged up or down the red point moved in the same way but that dragging the blue point left or right caused the red point to move in the opposite direction. Whilst drawing their attention to the direction of movement, Robert did not mention that dragging the blue point causes the red point to move the same distance, thus he did not draw his pupils attention to the geometric property that length is preserved under reflection.

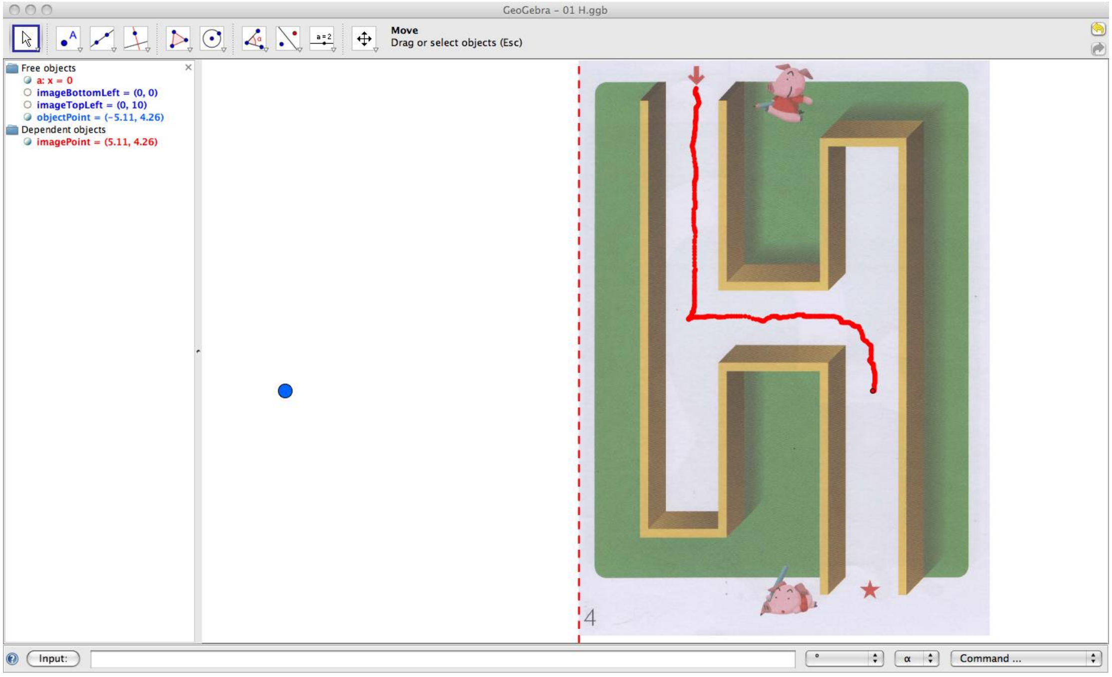


Figure 5.2 Robert's first GeoGebra maze with vertical line of reflection

Robert then displayed a maze with a horizontal line of reflection and, employing his *predict-then-test* strategy, asked the pupils to predict whether the relative direction of movement would be the same or different. The pupils correctly predicted it would change: now, dragging the blue point left or right would result in the red point moving in the same way but dragging the blue point up or down would cause the red point to move in the opposite direction. Contrasting these diagrams made the point that the relative direction of movement of the red and blue points was connected to the orientation of the line of reflection. At this juncture, Robert could have introduced the mathematical terms parallel and perpendicular to specify the nature of the connection between the relative direction of movement and the orientation of the line of reflection, thus generalising to state the effect of reflection on direction. He could also have noted that in both maze diagrams, independent of the orientation of the line of reflection, dragging the blue point causes the red point to move the same distance, hence length is preserved under reflection.

Robert did not introduce the mathematical terms parallel and perpendicular at this point nor did he note the geometric property that length is preserved under reflection. Instead, apparently on impulse, he offered his pupils a new challenge: to find out whether turning the mouse back to front would help them to complete the mazes, presumably by double-reversing the direction of movement. This challenge risked distracting from the aims of the lesson, since turning the mouse back to front involves a rotation of 180 degrees and not a reflection – it is an unhelpful *deviation from the lesson agenda* (Rowland et al., 2005). Later in the post-observation interview, Robert explained that it was "just something he was thinking about on the way in", dismissing it as "just a silly question to get a few of them thinking" [Rob-CS-int, 13.6.2012]. However, in asking this question, he missed an opportunity to capitalise on his pupils' correct predictions to generalise their maze-solving strategies towards a shared, formal understanding of the geometric properties of reflection.

The second opportunity occurred at the end of the lesson. Due to the shutdown of the computer system, the students were unable to begin the second GeoGebra activity Robert had prepared. After spending some time wrestling with the technology, Robert gave up and gathered the pupils to summarise the lesson. In this moment of contingency, Robert was inspired to ask his pupils to imagine the join between two rectangular tables, where they met along their longest edge, was a mirror. One of the pupils sitting at the table was holding a ball: this became the de facto 'blue point'. Robert discussed moving the 'blue point' close to the mirror, through the mirror (which he noted you can't do in reality), and finally *parallel* to the mirror. He did not have another chance to discuss what happens when the 'blue point' moves perpendicular to the mirror nor to discuss the preservation of length under reflection because, at that point, the bell rang for the next lesson.

Although his second opportunity to elicit the geometric properties of reflection was cut short, in the post-observation interview, when asked what he wished to do had there been more time, Robert did not articulate that he meant to discuss what happened when the blue point moved perpendicular to the line of reflection and to note that distances remained the same under reflection. These missed opportunities, together with the post-observation interview, suggest that Robert had not planned precisely what and how he would use mathematical terminology in his interventions to transform his pupils' maze-solving strategies into more formal understandings of reflection to connect with the aims of the lesson. In addition, when asked what he would have done differently in preparing the lesson, he focused solely on planning to prevent the technical difficulties arising rather than suggesting he could have been more precise in his use of mathematical terminology. Although Robert did not have much time to deliberate over the lesson (as the author has) and it is understandable that the technical difficulties that were so disruptive were uppermost in his mind, this suggests his experience during the lesson did not prompt Robert to recognise the need to plan his interventions more precisely to connect his series of maze activities with the mathematical aims of the lesson. In particular, Robert appeared to lack a frame of reference to help him identify *what* his mathematical difficulties were in using technology to make his pupils' tacit understandings explicit and, as a result, *why* his interventions appeared unsatisfactory. As noted in Chapter 1 and the introduction to this chapter, this is part of mathematical knowledge for teaching using technology, since in this study such knowledge is assumed not only to be a matter of knowing how – being competent in teaching mathematics using technology - but also of knowing what and why (Shulman, 1986, p.13).

Despite his favourable conception of technology, using the maze activities in practice was not trivial and Robert did not entirely succeed in making explicit the mathematical relationships the pupils were exploring using the GeoGebra software. This suggests that a positive stance towards technology, in terms of global aspects of teacher knowledge (e.g. Bowers & Stephens, 2011; Zbiek & Hollebrands, 2008), may not be sufficient to ensure a teacher's use of technology enhances mathematical instruction. The missed opportunities to transform pupils' maze-solving strategies into more formal statements of the geometric properties of reflection, using precise mathematical terminology to make connections between the maze activities and the aims of the lesson, suggest that mathematical knowledge for teaching using technology has a significant role to play in successful technology integration. Thus, whilst highlighting the role of teachers' conceptions in technology integration is important, this section has argued that the significance of mathematical knowledge for teaching using technology should not be overlooked nor underestimated.

5.2 The nature and content of the central TPACK construct

The analysis presented in this section focuses on exemplifying the dyadic construct TCK, using data from the four case study teachers' GeoGebra interviews, as a means of exploring the nature of the central TPACK construct. This section presents four examples of TCK arising from the GeoGebra interview data. These are listed below as follows, thus TCK is knowing about:

- rounding errors in measuring angles in GeoGebra;

- how angles are defined and measured in GeoGebra;

- issues of dependency; and

- dynamic imagery or preservation of geometric relationships under drag.

These examples are intended as being indicative rather than an attempt to provide an exhaustive list of the types of TCK that arose during the case study teachers' interaction with the GeoGebra file in interview. Using the examples presented below, this section argues that the central TPACK construct does not represent the integration of TCK with pedagogic knowledge. Instead, what distinguishes TPACK from the dyadic construct TCK appears to be mathematical knowledge, abstract in the sense that it generalises across particular technological contexts and mathematical topics. Nevertheless, it is argued that each example of TCK is itself an indicator of an example of TPACK. In particular, identifying knowledge as an example of TCK indicates the case study teachers' emphasis on technology and mathematical content in their articulated knowledge-in-action when addressing a situation involving a synthesis of mathematical, pedagogical and technology knowledge. This approach acknowledges the criticisms of the weak theorisation of the TPACK framework (Graham, 2011; Ruthven, 2014; Voogt et al., 2012) that draw attention to its ambiguities and lack of clarity in defining the framework's constructs and the inter-relations between them. In addition, it serves to highlight the *situated* nature of TPACK, as a transformation (Shulman, 1987; Rowland et al., 2005) of mathematical knowledge for the purposes of teaching using technology. Similar arguments can be made in relation to the other two dyadic constructs PCK and TPK, however they have not been included here for the sake of brevity. Finally, the apparent duality in seeing TPACK simultaneously as abstract, mathematical knowledge and yet as *situated* in the context of teaching using technology is addressed in the last section of this chapter, using Noss and Hoyles' (1996; & Kent, 2004; & Pozzi, 2002) notion of situated abstraction and Adler's (1999; 2001) concepts of the dilemma of transparency and visibility/invisibility.

Rounding errors in GeoGebra

At some point during the discussion of the three diagrams, each of the teachers appreciated that the software displayed rounding errors in measuring the angles at the centre and circumference of the circle (see Figure 5.3). The teachers noted the apparent breakdown in the doubling relationship between the angle at the circumference and the angle at the centre, but appreciated this was a result of rounding errors in the software's measurement rather than a counter-example that might lead them to reject the theorem. The appreciation of rounding errors indicates an understanding that the GeoGebra software does not conform to a mathematical ideal but merely models Euclidean geometry, subject to the limitations of computer hardware and programming. This is an example of TCK because it shows an understanding of how the GeoGebra software models geometric concepts and relations, in particular, how it affects the apparent relationship between the angles at the centre and circumference of the circle.

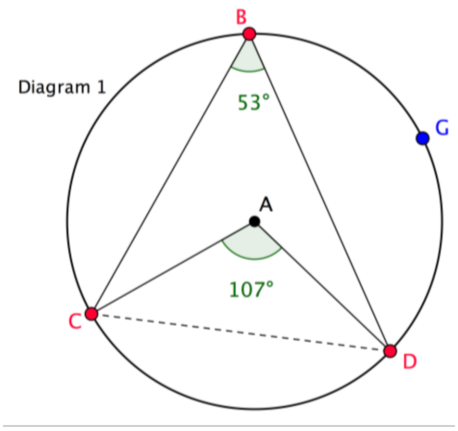


Figure 5.3 Rounding errors in angle measurement

In his initial dragging of the first diagram D1, Robert deliberately demonstrated the existence of rounding errors, thus exemplifying TCK. Later on, when prompted to discuss how he would address the issue of rounding errors with his pupils, Robert stated that trying to avoid this issue "I think you'll always be disappointed, so I think I would embrace it as quickly as you possibly can" (Rob-GGb-int, 13.6.2012). Robert explained that by 'embracing' this issue he meant he would deliberately introduce rounding errors as a possible instance of the conjectured relationship breaking down. As he explained, Robert illustrated how he embraces the rounding issue by setting GeoGebra to round numbers to 0 decimal places and then positioning D so that the angle at the circumference appears to be 58 degrees and the angle at the centre appears to be 117 degrees. As he did so, he stated, in this instance, it appears the conjectured circle theorem is "not true [...] but it really is true" (Rob-GGb-int, 13.6.2012). Thus recognising the rounding errors, he immediately dismissed them as irrelevant in the context of Euclidean geometry by rejecting the supposed counter-example. Robert went on to note that he uses the issue of rounding errors as a means of indicating the limitations of the computer hardware and software and that, although not a "great motivator", the issue of rounding errors provides a useful justification for the necessity of proof. This demonstration was very similar to his initial dragging of D1, yet here it appeared as TPACK: part of an articulated strategy for dealing with the issue of rounding errors in GeoGebra for the purposes of mathematics pedagogy.

Edward was deeply frustrated by the presence of rounding errors, as the quotes from his GeoGebra interview below indicate:

> Cos it's built for a purpose this [software]. The rounding really gets in the way of what you're trying to show.
>
> [later in the interview] I'd definitely mention it because sometimes it doesn't seem to work does it? ...it was one degree out... but I really see it as a hindrance to learning what's going on. I'd just, I'd have to keep saying 'Look, within rounding error this result is...', sort of, it's much less convincing... [Ed-GGb-int, 20.6.2012]

His frustration stems from the conflict between the 'real' world, where computer hardware places limits on the accuracy of measurement in GeoGebra, and the idealised mathematical world of Euclidean geometry to which the topic of circle theorems belongs. Thus the rounding errors are an indication to Edward that the software does not entirely 'fit' his purpose of providing a true representation of Euclidean geometry. Edward was also adamant that he would raise the issue of rounding errors but, unlike Robert, he did not articulate a pedagogic strategy to deal with the issue beyond drawing it to his pupils' attention.

Michael and Anne also suggested that they would raise the issue of rounding errors with their pupils. However, in addition, they both indicated that the discussion of rounding errors would provide an opportunity to link the hypothetical lesson on circle theorems more widely with topics on the accuracy of measurement. Michael suggests:

> It brings up a wider point of accuracy I guess and how everything is measured to varying degrees of accuracy and the importance of accuracy, because if you're not accurate then the theorems won't work. I guess if you're kind of sloppy in your angle measuring then you won't be able to prove anything. [Mic-GGb-int, 31.5.2012]

Similarly, Anne suggests linking the issue of rounding errors to the topic of upper and lower bounds of measurement, asking pupils within what bounds the angle could have been, given that it had been rounded to a certain degree:

> Yeah I would discuss it with pupils you know, the numbers were rounded. Yeah. I would discuss it with them, and depending on if we've done ... what is it called ... depending on if we've done bounds of measure, I could just bring it up, what could it [the angle] have been. [Anne-GGb-intB, 29.6.2012]

The approach Michael and Anne suggest risks confusion by conflating the real world with the idealised mathematical world of Euclidean geometry. Contrary to Michael's suggestion, in Euclidean geometry, the accuracy of angle measurement is immaterial to proof or whether the circle theorems 'work', precisely because they only exist in an idealised world where angles and other 'measures' are assumed to be exact.

Robert's articulation of his strategy for dealing with the issue of rounding errors in GeoGebra for the purposes of mathematics pedagogy is an example of TPACK. However, his strategy does not appear as a result of integrating pedagogic knowledge with TCK. Indeed all four teachers recognise the issue of rounding errors requires some kind of pedagogic intervention, however their strategies in response to this recognition differ. Edward did not articulate a pedagogic strategy to deal with the issue beyond drawing it to his pupils' attention. Michael and Anne risked confusion by connecting topics associated with the real world of measurement to the topic of circle theorems and proof, which exist only in the idealised mathematical world of Euclidean geometry. In contrast, Robert's strategy stands apart because it makes a clear mathematical distinction between these two worlds. He connects rounding errors to real world limitations of computer hardware and to using GeoGebra to establish conjectures though empirical testing, thus distinguishing circle theorems and proof as belonging to another ideal world of mathematics. Making this distinction requires mathematical knowledge: understanding that rounding errors and notions of accuracy are incompatible with Euclidean geometry and proof. Thus, Robert's articulation of his strategy for dealing with the issue of rounding errors in GeoGebra suggests that, in this case, TPACK is not the integration of TCK with pedagogic knowledge. Instead, TPACK appears to be mathematical knowledge, abstract in the sense that it generalises across particular technological contexts and mathematical topics. For example, rounding errors appear in measurement whenever one tries to relate mathematic ideals to the empirical world, whether in the context of using GeoGebra or a paper-and-pencil environment.

On the other hand, this knowledge appears to be simultaneously *situated* in the context of using GeoGebra to teach circle theorems. For example, knowing that the appearance of rounding errors in GeoGebra is an issue that requires a pedagogic strategy, e.g. drawing pupils' attention to rounding errors or otherwise, seems situated in the context of teaching. In addition, circle theorems are typically presented in textbooks through figures that invite the reader to imagine ideal mathematical objects i.e. where rounding is not an issue. Of course, as noted above, GeoGebra is not the only technological context where rounding errors arise. However, the combination of the textbook-style 'neatness' of GeoGebra figures and empirical measurement, appears to make the issue of rounding errors more salient - surprising even, given Edward's expression of frustration. Thus, Robert's articulation of his strategy for dealing with the issue of rounding errors in GeoGebra also suggests that TPACK is a synthesis of mathematical, pedagogical and technological knowledge, highlighting its situated nature. In this sense, TPACK appears to draw attention to a transformation of mathematical knowledge for the purposes of teaching using technology.

Defining angles in GeoGebra: Edward's dilemma

Michael and Edward were both prompted to question how angles are defined for the purposes of measurement in the GeoGebra software by unexpected configurations of D1 appearing during dragging, displaying the 'incorrect' angle at the centre (see Figure 5.4). After experimenting by dragging points C and D, and in Edward's case some prompting by the interviewer, they concluded the angle measured at the centre was dependent on the relative position of points C and D. More specifically, in GeoGebra the angle measured at the centre in D1 is defined by specifying the ordered triad of points CAD and measured anticlockwise from the line segment AC to the line segment AD. Thus when the relative positions of C and D are reversed, as in Figure 5.4, the angle appears to 'flip' between being less than 180 degrees and being reflex.

D1 had been designed so that, whilst the angle at the centre could become reflex, the angle measured at the circumference was constrained to be less than 180 degrees whatever the relative position of points C and D. Hence the 'correct' angle at the circumference in relation to the circle theorem was always displayed, however some configurations of D1 displayed the 'incorrect' angle at the centre. Michael and Edward's questioning of how the software defines and measures angles and their realisation of the angle at the centre's dependence on the relative positions of C and D is an example of TCK because it shows a developing understanding of how the GeoGebra software models geometric concepts and relations.

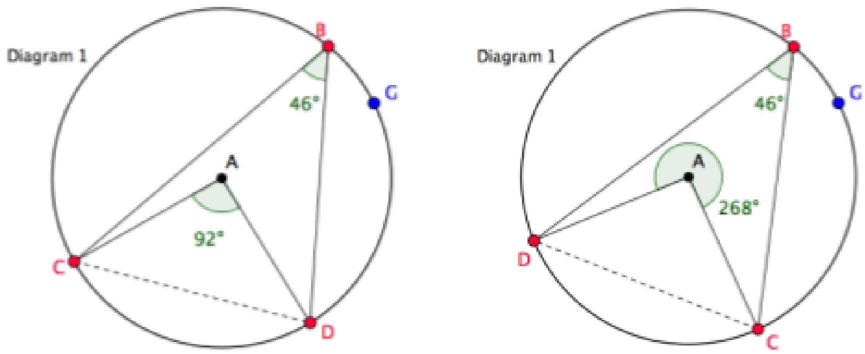


Figure 5.4 Angle measurement and reversing the relative positions of C and D

For Edward, the software's definition and measurement of angles was another source of frustration, appearing idiosyncratic in the way D1 'flipped' between displaying the correct and the incorrect angle at the centre in relation to the circle theorem. He argued:

> … this is sort of a function of how the software works isn't it, rather than a … is that bringing out anything useful mathematically that … that's just a bit annoying the way it does that, isn't it? [Ed-GGb-int, 20.6.2012]

Implicitly here, Edward appears to be comparing the ease of defining angles when teaching circle theorems without digital technologies, e.g. marking angles on a diagram using a pen on a whiteboard, in a textbook diagram or with pencil and paper, to the difficulties he faced understanding how angles are defined in GeoGebra. Diagrams presented in software such as GeoGebra are constrained to follow the rules for defining angles that have been programmed into that piece of software. It seems unlikely that a dynamic geometry software package will have been programmed to define angles according to their relevance to circle theorems, thus it may not be possible to construct a diagram such that the correct angle at the centre is always displayed. One of the affordances of drawing diagrams without digital technologies is that the relevant angles of the circle theorem may simply be marked on a diagram with a brief stroke of a pen or pencil, without needing to consider how they are defined precisely. It is not that a precise definition of the angles does not exist or is not necessary in a paper-and-pencil environment, of course, but that often it does not appear necessary to give it explicit consideration. This argument serves again to highlight the situatedness of individual teacher knowledge in the technological context.

A case where it might be necessary to give explicit consideration to a precise definition of the angles, even in a pencil-and-paper environment, would be when giving a full statement of the circle theorem, rather than an abbreviated form such as 'the angle at the centre is double the angle at the circumference'. For example, a full statement of the circle theorem is ‘the angle subtended at the centre by an arc is double the angle subtended at the circumference by the same arc’. The difference between the abbreviated form and the full statement is in the specification that the two angles must be subtended from the same arc. More specifically, using the full statement of the theorem clarifies which is the ‘correct’ and ‘incorrect’ angle at the centre.

In his initial discussion of D1, Edward assumes the angles are defined as being subtended by the chord CD:

> the chord C and D, joining C and D ... subtends an angle of 108 at the centre and 54 at the circumference, so uh … for this configuration of points it’s showing that the angle at the centre is twice the angle at the circumference. [...] ... so what it shows is the angle subtended at the circumference by chord CD is always twice the angle at the centre, irrespective of where B is. [Ed-GGb-int, 20.6.2012]

Defining the two angles as subtended from the chord is unproblematic as long as the two angles remain in the same segment; however, when they are in opposing segments the theorem appears to break down (see Figure 5.5 a and b).

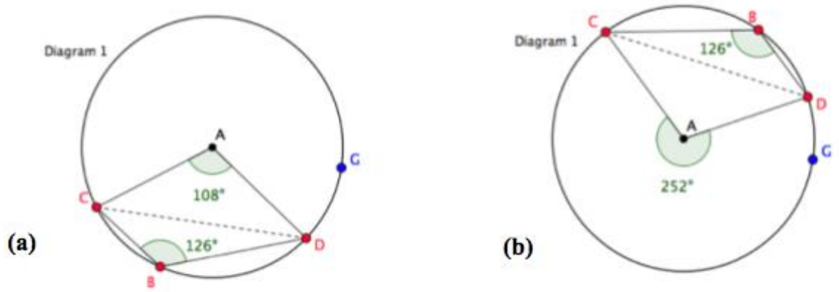


Figure 5.5 Angles in the opposite segments with **(a)** the ‘incorrect’ angle at the centre displayed and **(b)** the ‘correct’ angle at the centre displayed.

In fact, the doubling relationship between the angle at the centre and the angle at the circumference still holds when the angles are in opposing segments, as long as the angles are defined as being subtended from the same arc. Thus defining the angles based on the chord CD is an unfortunate narrowing of the cases to which the theorem may be applied.

The situation where the two angles appear in opposing segments occurred twice during Edward’s GeoGebra interview. Firstly, as depicted in Figure 5.5 (a), it occurred where the ‘incorrect’ angle at the centre is shown, assuming the angles in the circle theorem are defined as being subtended from the same arc. He had anticipated this case to some extent. Thus, for Edward, this case was not unduly problematic and did not disrupt his statement of the circle theorem defining the angles as subtended from the chord CD, as the quote below suggests:

> And then if you drag B this side [onto the minor arc CD], then suddenly it goes from 54 to 126. So ... uh ... what's happening there? So … uh … what's happening there is the angle on the other side of the 108 is now double the angle at the centre, the angle at the circumference … but it's not showing on the diagram, the computer's not showing that other angle ... but you can calculate it as 360 – 108, so 252. And 252 is double 126. Yeah. [Ed-GGb-int, 20.6.2012]

Instead he called this case a "complication", suggests "ignoring" it at least initially with pupils, and refers to the 'correct' angle at the centre, measuring 252 degrees, as "the reflex angle". His treatment of the case in Figure 5.5 (a) as a sort of deviant example or extension of his statement of the circle theorem, where the angle at the centre is reflex, avoided a mathematical critique of his definition of the angles being subtended from the chord. However, the situation arose for a second time, similar to Figure 5.5 (b), where the 'correct' angle at the centre is shown, assuming the angles in the circle theorem are defined as being subtended from the same arc. This time, the situation was unexpected and troubling for Edward. In particular, it leads him to question his previous definition of the central angle as being subtended by the chord CD. The following quote indicates his struggles as he attempted to find a correct mathematical interpretation of this configuration of D1, see Figure 5.6 for the numerical example he discusses at the start:

> Um ... so … let's take an example ... so 94 doubled is 188, so it's still true that … so that angle is twice that angle. But uh … how do you know it was that angle … so the computer is kind of showing you the right angle for what it's working for isn't it? But in words, how do you explain what that angle is, it's not really the angle that chord CD is subtending at the centre is it? Because it's that … chord CD is subtending that angle at the centre, so suddenly you have to say it's the other angle, the reflex angle at the centre that's subtending. So... so CD is subtending 99 at the circumference and, ... er ... the reflex angle is 198 yeah. Uh … which is not a very good explanation. [E laughs] [Ed-GGb-int, 20.6.2012]

At the end of this quote, Edward tries to re-state the theorem using a particular numerical example, taking into account his realisation that the 'correct' angle at the centre was not, as he previously assumed, the angle subtended by the chord CD. He struggles, eventually settling for "the reflex angle", whilst acknowledging this seemed inadequate.

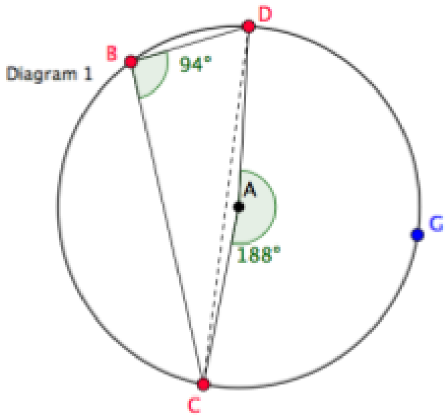


Figure 5.6 Edward's angle definition dilemma

Returning to Edward's frustration at the apparently idiosyncratic way GeoGebra defined and measured the angles in D1, the discussion above shows that instead of being "just a bit annoying" [Ed-GGb-int, 20.6.2012], the way GeoGebra defines and measures angles *does* bring out something mathematically useful. The variation in whether the 'correct' or 'incorrect' angle is displayed in D1 provides a means of discussing how angles are defined in other contexts and, in particular, how the angles referred to in the (abbreviated) 'angle at the centre is double the angle at the circumference' circle theorem are defined precisely in a full statement of the theorem. In addition, the reason why Edward's statement of the angle at the centre theorem, defining angles as subtending from the chord CD, is unsatisfactory relates to another circle theorem, commonly abbreviated to 'angles in the same segment are the same'. To show the geometrical connection between the two theorems, the latter might more helpfully be stated as 'angles subtended by the same arc are the same'. Articulating a strategy to use the way GeoGebra defines angles to raise these issues for the purposes of teaching circle theorems would be an example of TPACK. Again, such a strategy would not appear to depend on integrating pedagogic knowledge with TCK. Instead it requires mathematical knowledge regarding the precise definition of the angles in a full statement of the angle at the centre circle theorem, as well as how this theorem connects to other circle theorems. Hence, TPACK appears to be mathematical knowledge, abstract in the sense that it generalises across particular technological contexts and mathematical topics. For example, a precise definition of the angles in a full statement of the angle at the centre circle theorem should hold both in the context of using GeoGebra or a paper-and-pencil environment.

However, once again TPACK appears simultaneously to be mathematical knowledge *situated* in the context of teaching using technology. The issue of how angles are defined appears more salient and, like rounding issues, even surprising – for Michael and Edward at least – in the context of GeoGebra. In addition, the author notes that at the time, in England, the high-stakes GCSE examinations only required pupils to state an abbreviated form of the circle theorems. As a result, it is possible that the case study teachers were unaware of a precise definition of the angles in a full statement of the angle at the centre circle theorem. Hence, an individual teacher's TPACK may also be seen as situated in the examination system and national curriculum of the country in that teacher is working. This argument suggests that TPACK is a synthesis of mathematical, pedagogical and technological knowledge, highlighting its situated nature as a transformation of mathematical knowledge for the purposes of teaching using technology.

It should be noted here that Edward was not alone in his confusion over the definition of the angles in the angle at the centre theorem. Analysing his interview, the author realised that she did not know a full statement of the angle at the centre theorem. Indeed, the design of D1, incorporating the extraneous chord CD, indicates the author's own lack of knowledge in this regard. Fortunately, it proved useful in terms of data collection and triggered a growth in understanding for the author in what had previously seemed a fairly straightforward topic. None of the other case study teachers provided a full statement of the theorem. Whilst they each provided some form of the abbreviated version of the angle at the centre theorem, Edward was the only one to attempt to define the angles precisely. Although Robert briefly questioned the presence of the chord, he later appeared to find it useful to identify the relevant angles in a particularly contorted configuration of D1 and, alluding to the angles in the same segment theorem, he said:

> I think dragging point B kind of demonstrates the, I mean, the invariance of the fact that well it's still coming from the same chord. [Rob-GGb-int, 13.6.2012]

Trying to account for her own lack of knowledge regarding the full statement of the theorem, as the author noted in the previous paragraph, in England the high-stakes GCSE examinations only require pupils to state an abbreviated form of the circle theorems. Taking into account reports of widespread teaching-to-the-test (Ofsted, 2008), it is just possible that a generation of teachers and pupils have not been exposed to a full statement of the circle theorem other than by serendipitous experience. A quick Internet search reveals that on websites, such as the MyMaths website and the BBC GCSE Bitesize revision website, emphasise the abbreviated version of the theorem, whilst a precise definition of the angles via a full statement of the theorem is not provided.

Issue of Dependency in GeoGebra

Like other dynamic geometry software, GeoGebra introduces an explicit order of construction, a sequential organisation of actions necessary to produce a figure (Jones, 2000). Since certain parts of a figure are dependent on other parts, figures need to be created in a particular sequence, thus initiating a *hierarchy of dependencies* (Hölzl et al, 1994). For example, in relation to the dynamic geometry package *Cabri Geometre*, Hölzl et al (1994) describe three different types of points that occur. A *basic point* can be dragged freely without restriction. A *point on an object*, such as a circle or a line, can only move along the object since it is dependent upon that object for its existence. A *point of intersection* cannot be dragged at all: it can only be moved as a result of dragging the objects upon whose intersection the point depends. Hölzl (1996) argues that since these distinctions do not occur in a paper-and-pencil context, they could be considered as a form of geometry specific to Cabri. The same argument can be made for points in GeoGebra.

Each of the case study teachers showed TCK by demonstrating some understanding of the issue of dependency, when asked directly about the difference between the red and blue points in relation to all three diagrams. For example, they all noted that the red points are restricted to move on the circumference, whereas the blue points P and R, in D2 and D3 respectively, may be dragged freely. Hence, using the language above, P and R are *basic points*, whereas the red points are *points on an object*. More subtly, the circumference of the circle is constrained to go through point G in D1, so although it might initially appear that G is restricted to move on the circumference like the red points, in fact it is the circle whose movement is restricted. Thus point G is also a basic point, which may be dragged freely, and is therefore coloured blue. The case study teachers struggled during the interview to provide a plausible explanation for the colouration of point G.

Discussing the three circle theorem diagrams, the case study teachers mentioned issues of dependency less frequently in comparison to the *Practice Squares* GeoGebra file, where they were asked to practise ‘thinking aloud’ as they manipulated two diagrams, one constructed as a geometric *figure* of a square and the other as a *drawing* of a square, which deformed under drag. This is unsurprising since the *Practice Squares* file was designed as an introductory file to highlight the ‘rules of construction’ or dependency in

GeoGebra. Similarly, issues of dependency arose more frequently during the discussion of circle theorem diagrams 2 and 3 in comparison to D1. Nevertheless, Robert's initial comments about D1 indicate he is aware of the 'rules of construction' or issues of dependency in GeoGebra. He noted that the points B, C and D "are presumably all fixed to the circumference of the circle" (Rob-GGb-int, 13.6.2012). Similarly, with D2, before dragging the diagram, Robert immediately said, "I guess the big difference is P is no longer fixed" (Rob-GGb-int, 13.6.2012). Edward also recognised issues of dependency. Initially, describing D1, he implicitly referred to issues of dependency, noting "So in this diagram you've got three red points on the circumference" (Ed-GGb-int, 20.6.2012). With D2, he was more explicit, noting that "L's confined to the circumference" (Ed-GGb-int, 20.6.2012), then after further dragging, he generalised that all red points in the GeoGebra file are confined to move on the circumference but that P is "pretty free form" (Ed-GGb-int, 20.6.2012). Thus Robert and Edward's understanding of issues of dependency in GeoGebra demonstrate TCK i.e. an understanding of geometry *situated* in the software.

Dragging D1 provides a means of testing the conjecture that the position of point A at the centre and points B, C and D on the circumference is a *sufficient* condition for the angle at the centre to be double the angle at the circumference. Robert and Edward's opening comments regarding D1, describing the construction of the diagram, draw attention to the positioning of the points B, C and D. Their description suggests that the positioning of the B, C and D on the circumference is a critical feature of this particular diagram (as opposed to some other diagram). Thus, implicitly, they suggested any relationship that appears to hold between the angle at the centre and the angle at the circumference is conditional on the positioning of B, C and D on the circumference. Notably, although Michael and Anne demonstrate TCK, appreciating issues of dependency in later diagrams, they did not draw attention to the position of the points in D1 by describing the construction of the diagram, before indicating the doubling relationship between the angle at the centre and the angle at the circumference through dragging.

With D2, as Robert suggested in his discussion of the diagram, the status of point P as a basic point, released from the constraint of being positioned at the centre, allows an exploration of a partial converse of the theorem:

> I'd use something like this with pupils working independently and asking them questions like when are the angles the same? When are the angles twice one of the

other angles? And so on and establishing the conditions that are, you know, kind of working almost from the converse of the theorem. [Rob-GGb-int, 13.6.2012]

In particular, constructing P as a basic point, allows D2 to be used as means of testing the conjecture that positioning P on the centre is a *necessary* condition for the angle at P to be double the angle at the circumference, given that L, M and N are positioned on the circumference. Indeed, whilst dragging point P, Robert stumbled on a set of counter-examples to this conjecture, although during the interview he eventually rejected them, assuming they appeared to occur due to rounding errors. Using the trace function in GeoGebra, it is possible to indicate the locus of point P where the angle at P subtended by the arc MN is double the angle at the circumference subtended by the same arc – see Figure 5.7.

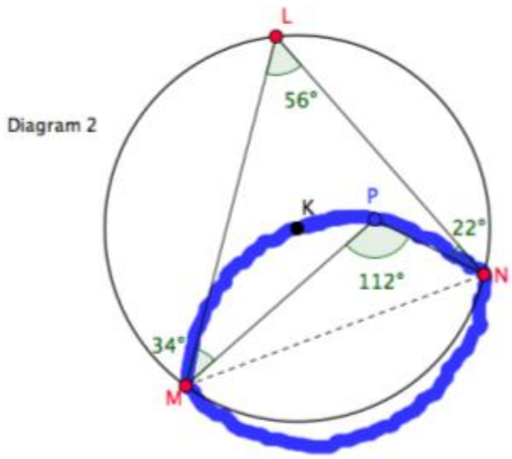


Figure 5.7 A trace indicating the locus of P where the doubling relationship between the angles at L and P holds, given points L, M and N are on the circumference.

Similarly, with D3, constructing point R as a basic point, allows the diagram to be used as a means of testing the conjecture that positioning R on the circumference is a necessary condition for the angle at the centre of the circle to be double the angle at R, given that points S and T are positioned on the circumference and point Q is positioned at the centre. In this case, the position of R on the circumference *is* a necessary condition for the doubling relationship to hold i.e. positioning R off the circumference causes the relationship to break down. Edward showed his appreciation of this purpose for D3 by dragging point R on and off the circumference to show the "negative case" of the angle at the centre theorem, as he termed the set of points where the doubling relationship breaks down. In particular, in relation to D3, he states:

E: it's really good for showing that the circle really is the special set of points where it works. So there's something beautiful about that. [Ed-GGb-int, 20.6.2012]

Michael also recognises the possibility of using D3 for the purpose of testing the necessity of positioning point R on the circumference:

> M: Um, I guess it's [D3 is] useful in that it shows that the rules only apply on the circumference or certainly that particular rule of the double angle at the centre, circumference etc.
>
> I: Mmm, mmm.
>
> M: Um, so I guess it's useful to show that that [D3] doesn't work, whereas that [D1] does work. [Mic-GGb-int, 31.5.2012]

Articulating a strategy for using the diagrams D1, D2 and D3 as a means of testing the positioning of the points on the circumference and at the centre as necessary and sufficient conditions for the angle at the centre circle theorem to hold, for the purposes of teaching the topic of circle theorems, is an example of TPACK. Thus Robert, Edward and Michael's recognition of the potential of diagrams D2 and D3 for testing partial converses of the circle theorem showed TPACK. Again, this suggests TPACK is mathematical knowledge, abstract in the sense that it generalises across particular technological contexts and mathematical topics. For example, knowledge of necessary and sufficient conditions and how they relate to the converse of a theorem should hold across particular technological contexts and mathematical topics.

However, this example also highlights the situated nature of an individual teacher's TPACK: that this knowledge is a function of the context in which the teacher is operating. For example, the issue of dependency itself depends on the 'rules of construction' programmed into the particular software package. In addition, issues of dependency did not arise as frequently with the circle theorem diagrams as they did with the Practice Squares file, neither did they occur as frequently with D1 as they did with D2 and D3. Finally, as Robert alludes in the quote below, knowledge of the converse of circle theorems or of their necessary and sufficient conditions is not included in GCSE examination syllabi (e.g. EdExcel, 2012), nor is it part of the English National Curriculum (QCA, 2007) for mathematics.

> I: Would that be something that you highlighted to kids, the converse?
>
> R: I have in the past yeah, but only with very, very strong groups. You know because we don't do it much do we really? I mean I think the only thing they really need to, I think, you know, in our kind of curriculum I think, [...] … we tend to use these theorems just in one direction. [Rob-GGb-int, 13.6.2012]

Thus an individual teacher's TPACK may be seen as a function of the software, the mathematical topic and even the particular diagram the teacher is using, as well as the examination system and national curriculum of the country in that teacher is working. Highlighting the situated nature of TPACK suggests it is a synthesis of mathematical, pedagogical and technological knowledge, representing a transformation of mathematical knowledge for the purposes of teaching using technology.

Dynamic imagery or preservation of geometric relationships under drag

A fourth example of TCK is an understanding that geometric relationships are preserved under drag, thus the dynamic imagery of GeoGebra is a means of representing geometric relationships. All four teachers demonstrated an understanding that geometric relationships are preserved under drag, by moving points B, C and D to different positions around the circumference and noting that the relationship between the angle at the centre and the angle at the circumference was preserved. Demonstrating D1 to pupils, Robert, Edward and Anne each suggested they would begin by dragging point B, showing initially that the relationship holds wherever B was placed on the major arc CD, before arguing that it also holds if B is positioned on the minor arc CD. Thus, by dragging B, they demonstrated the angle at the centre is double the angle at the circumference wherever B is positioned on the circumference – although none of the teachers entirely resolved what occurs when B coincides with either point C or D. Robert later clarified that dragging B also demonstrates the theorem that angles in the same segment are the same. In contrast to the other three case study teachers, when asked how he would demonstrate D1 to pupils, although Michael briefly dragged B on the major arc CD, he did not position B on the minor arc before continuing his exploration of D1 by dragging points C and D.

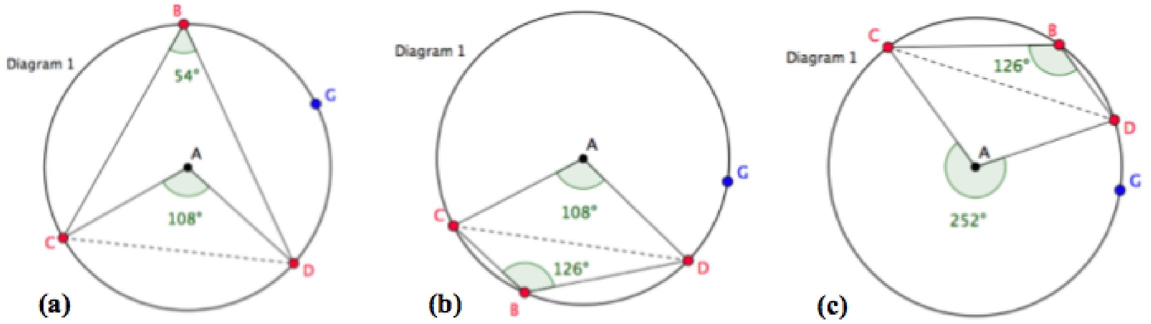


Figure 5.8 (a) the ‘arrowhead’ configuration **(b)** the ‘convex quadrilateral’ configuration produced by dragging B onto the minor arc **(c)** the convex quadrilateral configuration produced by dragging points C and D upwards

Dragging B first, resulted in Robert, Edward and Anne emphasising the ‘arrowhead’ configuration of D1 (see Figure 5.8 a) as the normal or standard configuration of the angle at the centre theorem. As a consequence, when they dragged B onto the minor arc CD, the ‘convex quadrilateral’ configuration of D1 where the angle at the centre is reflex (see Figure 5.8 b) appeared to be an abnormal or non-standard configuration, requiring a special statement and explanation – exacerbated perhaps by the presence of the extraneous chord. In particular, as Robert later noted, dragging B demonstrates the theorem that angles in the same segment are the same. Thus dragging B onto the other, minor segment causes this theorem to break down, distracting attention from the fact that the angle in the centre theorem still holds. The distraction caused by moving B onto the minor segment also seemed to disrupt the case study teachers’ knowledge of the angle at the centre theorem as holding true for all positions of B, C and D, as long as the three points remain distinct. For example, Robert said “I can’t remember what happens if I bring it over here” (Rob-GGb-int, 13.6.2012) before dragging B onto the minor arc. He notes subsequently that previously he has prevented B from being dragged onto the minor arc, thus barring the convex quadrilateral configuration, “Because on diagrams I’ve had in the past I’ve forced it to just lie on the major arc” (Rob-GGb-int, 13.6.2012). Anne struggles with whether the angle at the centre theorem has broken down or not when B moves onto the minor segment, she says: “the rule has not fallen apart ... it hasn’t fallen apart here, in that um ... yeah the rule has fallen apart a bit, hasn’t it?” (Anne-GGb-intA, 28.6.2012). Whilst Edward is convinced that the theorem still holds when B is positioned on the minor arc, he refers to this configuration as a “complication” (Ed-GGb-int, 20.6.2012) and later is confounded when he realises that the configuration is not encompassed by his statement of the theorem.

In Michael's case, the convex quadrilateral configuration *first* arose as shown in Figure 5.8 (c) as a result of dragging C and D, rather than encountering the configuration as shown in Figure 5.8 (b) as a result of dragging B onto the minor arc. Arising in this manner, the convex quadrilateral configuration does not appear as an abnormal or non-standard configuration. Instead it appears as a natural consequence of varying the angles at the centre and circumference to observe the doubling relationship. Thus the convex quadrilateral configuration appears on more equal terms with the arrowhead configuration, as one among a number of cases of the angle at the centre theorem. At least the convex quadrilateral configuration does not trouble Michael in the same way as the other three case study teachers. When it first arises as a result of dragging points C and D 'upwards', Michael remarks that the rule still applies, hesitating only to point out rounding errors:

> M: ... Then I can start moving C and D to probably you know nice round numbers maybe, so 120 so you can see it's twice as much. If you go … yeah, I'd probably then eventually move it up like that to form like a quadrilateral so that the rule still applies.
>
> I: Mmm, mmm.
>
> M: Well it was an arrowhead before, wasn't it, but now it's a reflex angle on the outside of a quadrilateral but it still applies just. I guess that's a rounding [error]. We were talking about that before weren't we? [Mic-GGb-int, 31.5.2012]

After this initial exploration, dragging C and D, Michael concluded:

> Um, so yeah I think that's about as much as I would do with diagram 1. In fact I have done something similar with a similar diagram before. [Mic-GGb-int, 31.5.2012]

Later in the interview, he was prompted to drag point B by the author. He then dragged point B past D and onto the minor arc CD causing the convex quadrilateral configuration to arise as shown in Figure 5.8 (b). However, unlike the other three case study teachers, this configuration seemed to cause him no confusion. He simply noted that he had mentioned this configuration earlier:

> .... and then this goes … well you know I mentioned that one earlier didn't I ... [Mic-GGb-int, 31.5.2012]

Given the difficulties Robert, Anne and Edward faced interpreting the convex quadrilateral configuration when it first arose as a result of dragging B onto the minor arc CD, it would seem reasonable to assume that pupils might also find this configuration a cause for confusion. In addition, dragging point B first seems to place undue emphasis on the arrowhead configuration. Thus it seems preferable to adopt a strategy of initially dragging points C and D to manage the dynamic imagery so that pupils first encounter the convex quadrilateral configuration as shown in Figure 5.8 (c), thereby also maintaining this configuration on more equal terms with the arrowhead configuration. Articulating such a strategy for managing dynamic imagery is an example of TPACK. This requires an understanding that dragging point B in D1 confuses the angles in the same segment theorem with the angle at the centre theorem, and thus may be unwise as an introductory dragging strategy (to either theorem). Hence such a strategy requires controlling mathematical variation to highlight the specificity and the generality of a particular geometric relationship (rather than any other geometric relationship), in this case, the relationship that the angle at the centre of circle is double that at the circumference. In this sense, TPACK appears to be mathematical knowledge, abstract in the sense that it generalises across particular technological contexts and mathematical topics. For example, the codes *choice and use of examples* and *decisions about sequencing* in the Knowledge Quartet (Rowland et al., 2005) represent instances of controlling numerical and geometric variation that are intended to apply across technological contexts and mathematical topics. However, the means of controlling mathematical variation through *dragging* to generate examples is particular to dynamic geometry software. In addition, Leung and Lee (2013) suggest dynamic variation is central to software such as GeoGebra, hence controlling variation to highlight the specificity and the generality of a particular geometric relationship may be more salient in these technological contexts. Again this serves to highlight the situated nature of TPACK as a transformation of mathematical knowledge for the purposes of teaching using technology.

5.3 Conclusion: the nature and content of mathematical knowledge for teaching using technology

The analysis of case study teacher Robert's computer suite lesson, presented in section 5.1, suggested that a positive stance towards technology, in terms of global aspects of teacher knowledge (e.g. Bowers & Stephens, 2011; Zbiek & Hollebrands, 2008), may not be sufficient to ensure a teacher's use of technology enhances mathematical instruction. In addition, this analysis suggested that using technology places significant demands on individual teachers' knowledge.

Further, the analysis of the four case study teachers' GeoGebra interviews in section 5.2 suggested that the nature of mathematical knowledge for teaching using technology, as represented by the central TPACK construct, is not the integration of the dyadic constructs, TCK, PCK or TPK, with a third domain of knowledge. For example, in the exemplars of TCK described in section 5.2, the central TPACK construct did not appear to depend on integrating pedagogic knowledge with a combination of technological and mathematical knowledge. Instead, these exemplars of TCK highlighted mathematical knowledge for teaching using technology as *mathematical* knowledge, abstract in the sense that it generalises across particular technological contexts and mathematical topics. For example, rounding errors appear in measurement whenever one tries to relate mathematic ideals to the empirical world, whether in the context of using GeoGebra or a paper-and-pencil environment. In this sense, viewing mathematical knowledge for teaching using technology as abstract *mathematical* knowledge draws closer to the integrative perspective (Graham, 2011) of TPACK in that such knowledge may not be seen as particular to the context of teaching with technology. On the other hand, the TCK exemplars simultaneously highlighted the *situated* nature of mathematical knowledge for teaching using technology. Thus, having an articulated pedagogic strategy for dealing with the issue of rounding errors in GeoGebra, for example, suggested that the central TPACK construct signified a synthesis of mathematical, pedagogical and technological knowledge, indicating the situated nature of mathematical knowledge for teaching using technology. In this sense, the central TPACK construct appears to draw attention to the transformation (Rowland et al., 2005, Shulman, 1987) of mathematical knowledge for the purposes of teaching using technology.

The apparent duality of mathematical knowledge for teaching using technology produces a tension: teacher knowledge appears simultaneously to be both situated in and abstracted across technological contexts and mathematical topics. To this extent, Noss and Hoyles' (1996; & Kent, 2004; & Pozzi, 2002) notion of *situated abstraction* seems appropriate as a description of mathematical knowledge for teaching using technology. This notion is useful in allaying the concern that seeing mathematical knowledge for teaching using technology as situated should not be taken to mean that each piece of technology requires teachers to learn an entirely new and distinct domain of mathematical knowledge for teaching i.e. mathematical knowledge for teaching using GeoGebra as distinct from mathematical knowledge for teaching using a static paper-and-pencil environment as distinct from mathematical knowledge for teaching using Geometer's Sketchpad and so on. Thus a teacher with mathematical knowledge for teaching using GeoGebra and mathematical knowledge for teaching using a static paper-and-pencil environment should find it easier to develop mathematical knowledge for teaching using Geometer's Sketchpad and so on. However, at the same time the notion of situated abstraction allows that mathematical knowledge for teaching using technology may not always be separated from the context of its construction or application, thus the particular technological context in which it was developed may be central to its meaning (Noss et al., 2002).

Adler's (1999) use of Lave and Wenger's (1991) notion of *transparency* to describe teachers' dilemmas in negotiating the dual *visibility* and *invisibility* of talk as a resource in the practice of school mathematics also appears useful although, here, it is applied to the use of technology as a resource in the practice of teaching school mathematics. Adler (1999) describes Lave and Wenger's use of the metaphor of a window to explain their notion of transparency:

> Lave and Wenger (1991) used the metaphor of a window to clarify their concept of transparency. A window's invisibility is what makes it a window. It is an object through which the outside world becomes visible. However, set in a wall, the window is simultaneously highly visible. In other words, that one can see through it is precisely what also makes it highly visible.

Thus technology as a teaching resource for mathematics needs to be simultaneously both visible, so that it can be noticed and used in the practice of teaching school mathematics, and invisible so that attention is focused on the subject matter, teaching *mathematics* to pupils. The specificity of teaching mathematics using a particular piece of technology influences the mathematics that can be taught. For example, sketch diagrams in paper-and-pencil environments are flexible in that they do not have to obey fixed rules in relation to measurement of lengths and angles. Diagrams in GeoGebra appear more rigid in this respect - hence Edward's irritation with rounding errors and the definition and measurement of angles in the software. The flexibility of the paper-and-pencil environment by contrast affords the user the freedom to imagine they are working in an ideal mathematical world, where perfect circles, exact angle measurement, circle theorems and proof 'exist'. Hence, whilst all four case study teachers were generally favourable towards using the GeoGebra diagrams for establishing the theorems on an empirical basis, Robert and Edward did not see the software as a means for developing proof. Similarly, the window frame, its shape and positioning on the wall, influences which part of the outside world can be seen. Thus teachers need to understand the significance of the particular technology for the mathematics they are teaching; hence the technology requires explicit attention, it needs to be visible. In this sense, mathematical knowledge for teaching using technology is always situated, since the technological context in which it is being applied is central to its meaning.

Simultaneously, however, technology should enable the *teaching of mathematics*, in this case the GeoGebra software should enable the teaching of circle theorems, and should thus be invisible. It is the window through which mathematical knowledge for the purposes of teaching can be seen: the GeoGebra software is a means of controlling numerical and geometric variation so that pupils are systematically exposed to a wide range of examples of the angle at the centre theorem. Here, mathematical knowledge for teaching using technology appears more abstract, allowing teachers to make comparisons and connections across technological contexts.

Adler's description of a dilemma of transparency where the teacher manages talk as a classroom resource, so that it is neither too visible for pupils, obscuring the mathematical subject matter, nor too invisible so that they are unable to access it, has some explanatory value for this study. However, here, the dilemma is managing technology so that it does not become too visible for teachers, obscuring mathematical knowledge for teaching using technology, nor too invisible, so that teachers assume that the use of technology unproblematically provides a pedagogic structuring for mathematics. For example, Edward's irritation with the definition and measurement of angles in GeoGebra indicated that the software was too visible for him. In this case the GeoGebra software obscured his access to mathematical knowledge for teaching using technology, i.e. the articulation of a pedagogic strategy to use the software as a means for discussing how to define the angles involved in a full statement of the angle at the centre theorem. On the other hand, the case study teachers' focus on dragging point B in D1, emphasising the arrowhead configuration as the standard configuration of the angle at the centre theorem provides an instance where technology seems too invisible. Here, the unintentional pedagogic structuring of mathematics suggests that the technology has become too invisible, with an assumption that technology provides unproblematic access to mathematical knowledge for teaching. Similarly, the Knowledge Quartet code *adherence to textbook* might suggest instances where technology in the form of a textbook has become too invisible in the practice of teaching school mathematics.

The TPACK framework and the central TPACK construct in particular were useful for the purposes of this study for concentrating attention on teacher knowledge in relation to technology, identifying mathematical knowledge for teaching using technology as a focus for research (see also Chapters 1 and 2). In a similar sense, Shulman's (1986) concept of PCK has been highly productive, stimulating research focused on the nature and content of teacher knowledge (see Chapter 2).

The central TPACK construct also provided a useful metaphor for recognising the nature of mathematical knowledge for teaching using technology simultaneously as abstract *mathematical* knowledge and as a synthesis of content, pedagogy and technology knowledge i.e. highlighting the *situatedness* of mathematical knowledge in the context of teaching and technology. In other words, the strength of the TPACK framework for this study is the contribution it makes in helping to manage the dilemma of transparency: the framework helps to balance the dual visibility and invisibility of technology as a resource in the practice of teaching school mathematics. The weak theorisation of the TPACK framework (Graham, 2011; Ruthven, 2014; Voogt et al., 2012) provides a key affordance in this respect, by allowing the user to move between an integrative and transformative perspective in particular in relation to the central TPACK construct.

However, the categorisation of knowledge, induced by the Venn diagram depiction of the TPACK framework, into the seven constructs (TK, CK, PK, TCK, PCK, TPK and the central TPACK construct) suffers from difficulties similar to those of other knowledge taxonomies (e.g. Ball et al., 2008). Namely, it seems unlikely that demonstrable progress will be made in establishing "persuasive and productive distinctions" (Ruthven, 2011) between the categories of knowledge hypothesised by the TPACK framework, since as the examples discussed in this chapter suggest, problems of using technology to teach mathematics generally cannot "be adequately framed in 'pure' terms drawn from a single knowledge domain, or even by drawing on several domains independently" (ibid). In addition, the abstract quality of the TPACK categories and their apparent separation from each other tends towards an impression of static 'knowledge', for which Shulman's original categorisation of PCK, on which the framework was based, has similarly been criticised (Fennema & Franke, 1992). Thus although a transformative perspective of the TPACK framework draws closer to Shulman’s (1987, also Rowland et al., 2005) use of transformation, nevertheless the central TPACK construct is theorised as a new *domain* of knowledge rather than as a dynamic *process*. Finally, the analysis presented in section 5.2 suggests that the content of mathematical knowledge for teaching using technology consists of a repertoire of articulated strategies for using the constraints and affordances (Greeno, 1998) of the technology for the purposes of teaching mathematics. However, beyond this the TPACK framework does not appear useful in terms of producing a more fine-grained analysis (Ruthven, 2014) of the content of mathematical knowledge for teaching using technology because it lacks subject-specificity. Thus, in Chapter 6, the Knowledge Quartet provides an analytic tool for a more detailed investigation of the content of mathematical knowledge for teaching using technology due to its development in the field of mathematics education research.

5.4 Summary

This chapter has focussed on individual teachers’ own knowledge in relation to using technology to teach mathematics. Thus this chapter has begun to address RQ2, by describing the nature of mathematical knowledge for teaching using technology in terms of the notion of situated abstraction and indications of content in relation to the topic of circle theorems. However, beyond this the TPACK framework did not appear useful in terms of producing a more fine-grained analysis (Ruthven, 2014) of the content of mathematical knowledge for teaching using technology because it lacked subject-specificity. Hence Chapter 6 returns to RQ2, using the Knowledge Quartet as an analytic tool to provide such a fine-grained analysis of the content of mathematical knowledge for teaching using technology in relation to the topic of circle theorems. In focusing on individual teachers’ own knowledge in relation to using technology to teach mathematics, neither Chapter 5 nor 6 include an explicit consideration of the participatory relationship between teacher and technology. This is addressed in Chapter 7, where Hutchins’ view of distributed cognition provides a framework for understanding how individual teachers’ knowledge is involved in the participatory relationship (Remillard, 2005) with technology.

Chapter 6 - Conceptualising mathematical knowledge for teaching circle theorems using technology

The literature review in Chapter 2 found that research on technology in mathematics education has paid relatively little attention to teachers' knowledge of specific mathematical concepts in relation to technology. By comparison, research on teacher knowledge in the field of mathematics education is relatively well conceptualised, although limited in terms of its curriculum coverage. For example, Ma's (1999) research focused on arithmetic operations and Askew et al (1997) focused on numeracy. In addition, research on teacher knowledge in the field of mathematics education has rarely considered teachers' mathematical knowledge for teaching in the context of technology use. Hence the literature review concluded that the nature and content of teachers' mathematical knowledge for teaching using technology remained an unresolved question, leading to the identification of Research Question 2:

RQ2 *What is the nature and content of teachers' mathematical knowledge for teaching using technology, as represented by the central TPACK construct?*

Chapter 5 began to address RQ2, using Noss and Hoyles' (1996; & Kent, 2004; & Pozzi, 2002) notion of situated abstraction and Adler's (1999; 2001) concepts of the dilemma of transparency and visibility/invisibility to describe the nature of mathematical knowledge for teaching using technology as abstract, mathematical knowledge and yet simultaneously as mathematical knowledge *situated* in the context of teaching using technology. The analysis presented in Chapter 5 using the TPACK framework suggested that the content of mathematical knowledge for teaching using technology consists of a repertoire of articulated strategies for using the constraints and affordances (Greeno, 1998) of the technology for the purposes of teaching mathematics. However, beyond this the TPACK framework did not appear useful in terms of producing a more fine-grained analysis (Ruthven, 2014) of the content of mathematical knowledge for teaching using technology because it lacked subject-specificity.

This chapter addresses RQ2 through a detailed investigation of the content of mathematical knowledge for teaching using technology in relation to the topic of circle theorems, arriving at a conceptualisation of this knowledge. This conceptualisation demonstrates the high level of complexity of mathematical knowledge for teaching using technology in relation to the topic of circle theorems. This high level of complexity is justified post-hoc by the analysis of data from the four case study teachers' semi-structured GeoGebra interviews, using the Knowledge Quartet.

The Knowledge Quartet was chosen as a suitable tool for analysis primarily due to the framework's development as a means of focusing attention on teachers' mathematical knowledge. The suitability of the framework for this study is discussed further in section 6.1 (see also Chapters 1 and 3). The analysis presented in this chapter demonstrates the emergence of *choice and use of examples* and *decisions about sequencing* as central codes, using exemplars of codes from each of the supra-categories of the Knowledge Quartet. The centrality of these two codes also provides a post-hoc justification for the development of a *circle theorem case list* for each case study teacher, as a means of measuring mathematical knowledge for teaching circle theorems using technology, allowing a comparison of both the specific circle theorem cases arising and the sequence in which they arose during the interview. Thus this chapter also addresses the subsidiary Research Question 2a:

RQ2a *How does a conceptualisation of teachers' mathematical knowledge for teaching using technology suggest ways in which such knowledge could be measured?*

The literature review in Chapter 2 found that relatively sophisticated measures of teachers' mathematical knowledge for teaching have been developed in research on teacher knowledge within the field of mathematics education. However, research on constructing measures of teachers' mathematical knowledge in teaching has only recently begun to explore the impact of cultural context (e.g. Delaney et al., 2008; Blomeke & Delaney, 2012). The impact of the methods with which test items are applied (e.g. multiple-choice questions, paper-and-pencil tests etc) and the mathematical topics covered on what is measured has received relatively little attention. In particular, the development of test items embedded in a technological context to measure teachers' mathematical knowledge for teaching appears to have been neglected.

The review of literature in Chapter 2 also suggested that research on mathematical knowledge for teaching emphasises the importance of connected knowledge. For example, Ma (1999) explains her notions of *breadth* and *depth*, which serve to characterise a *profound understanding of fundamental mathematics*, in terms of connections among concepts and between topics. Thus a reasonable hypothesis is that mathematical knowledge for teaching using technology might be measured in terms of the connectedness of teachers' mathematical knowledge, for example, the number of connections between concepts. In particular, from their concept-mapping interviews, Askew et al (1997, p.55) found a moderate relationship between their variable *depth*, the percentage of links that were explained by the teacher in conceptual terms, and pupil gains. Their other variables in relation to teachers' mathematical knowledge did not show any relationship to pupil gains. In addition, depth was the only variable which appeared to be related to teaching orientation, with transmissionist teachers making a relatively low percentage of conceptual links compared to their connectionist and discovery-oriented colleagues (Askew et al, 1997, p. 60). This suggests a measure of mathematical knowledge for teaching using technology, defined in terms of conceptual connections, might show a moderate relationship with teachers' transmissionist or connectionist orientation.

In section 6.3, this chapter concludes by tentatively suggesting ways in which the *circle theorem case list* could be used to measure teachers' mathematical knowledge for teaching using technology in terms of conceptual connections in relation to the topic of circle theorems. Validating these ways of measuring teachers' mathematical knowledge for teaching using technology in relation to the topic of circle theorems was beyond the scope of this study.

6.1 Analysis of GeoGebra interviews using the Knowledge Quartet

In terms of this study, the strengths of the Knowledge Quartet lie in the framework's focus on *mathematical knowledge* for teaching and the grounding of its codes in classroom observation, as prototypical classroom situations where mathematical knowledge for teaching arises, thereby maintaining strong face and content validity. In Chapter 5, the distinction between the dyadic constructs and the central TPACK was found to be in terms of mathematical knowledge, providing evidence that mathematical knowledge for teaching using technology can be thought of primarily as mathematical knowledge, albeit simultaneously situated in the context of teaching using technology. The TPACK framework does not provide a means for analysing mathematical knowledge due to its lack of subject-specificity. Thus, in relation to this study, the Knowledge Quartet's focus on mathematical knowledge was a useful attribute, providing a subject-specific lens through which to view the data, complementing the TPACK framework's focus on technology and compensating for its relative lack of subject-specificity. In particular, the Knowledge Quartet's Connection supra-category seems advantageous as an analytic tool in relation to the hypothesis that mathematical knowledge for teaching using technology might be measured in terms of the connectedness of teachers' mathematical knowledge. In addition, research on the importance of variation to structure sense-making (Marton & Booth, 1997; Watson & Mason, 2005; 2006) suggests the code *choice and use of examples* (Transformation) may be advantageous as a tool for analysing mathematical knowledge for teaching using technology in the context of the semi-structured GeoGebra interviews, since dynamic variation is central to such software (Leung & Lee, 2013).

On the other hand, the lack of focus on technology in the Knowledge Quartet, in particular on digital technologies, was a possible weakness in attempting to develop a measure of mathematical knowledge for teaching circle theorems using technology. However, as the literature review (Chapter 2) suggests, the Knowledge Quartet is not unique amongst frameworks on mathematical knowledge for teaching in lacking a focus on technology. The strong face validity of the Knowledge Quartet also made the framework generally appealing to the author. In addition, and perhaps more importantly, analysing the GeoGebra interview data using the Knowledge Quartet provided a means of ensuring that any measure of mathematical knowledge for teaching circle theorems using technology based upon this analysis would be likely to retain the strong content validity of the parent framework. However, the strong content validity of the Knowledge Quartet derives from the framework's grounding in classroom observations, thus applying its codes to interview data might result in some discordancy. These issues regarding the Knowledge Quartet are returned to later in Section 6.1.5.

The following sections draw on data from the discussion of the first diagram D1 in the semi-structured interviews on using GeoGebra files to teach circle theorems to exemplify the four supra-categories of the Knowledge Quartet, namely *Foundation*, *Transformation*, *Connection* and *Contigency*. These examples are intended as being indicative rather than an attempt to provide an exhaustive list of each Knowledge Quartet code that arose during the case study teachers' interaction with the GeoGebra file in interview. In particular, the examples have been chosen to illustrate the emergence of *choice and use of examples* (Transformation) and *decisions of sequencing* (Connection) as central codes. In addition, where appropriate, indications or examples are given where codes were absent or required an interpretation that appeared to deviate somewhat from the original Knowledge Quartet meaning due to the interview or technological context.

6.1.1 Foundation

The extraneous chord

D1 was designed as a means of demonstrating the angle at the centre theorem. All four of the case study teachers, Robert, Anne, Edward and Michael, readily recognised the purpose of the diagram. D1 was constructed with the angle at the circumference at point B, subtended by the arc CD, and the angle at the centre-point A, also subtended by the arc CD, as shown in Figure 6.1. In addition, an extraneous chord was constructed as a dashed line segment joining points C and D.

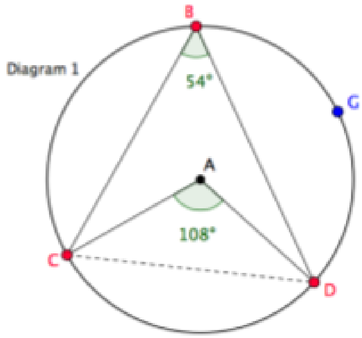


Figure 6.1 The starting configuration of Diagram 1

The chord CD is extraneous because it is not mathematically useful either in defining or proving the angle at the centre theorem. Indeed, the extraneous chord is a potential distraction from the angle at the centre theorem, since it divides the circle into two segments, drawing attention to another circle theorem: the angles in the same segment theorem. It could be argued instead that the chord CD provides a useful means of connecting the two theorems. However, the chord CD is also not necessary either in defining or proving the angles in the same segment theorem. In fact, such a connection might be better made mathematically by introducing the 'angles in the same segment' theorem as 'angles on the circumference subtended by the same arc'. Thus recognising the chord CD as extraneous to D1 for the purpose of demonstrating the angle at the centre theorem, or at least questioning its relevance, is an example of *overt subject knowledge*, indicating a coherent mathematical understanding of this circle theorem.

Robert was the only case study teacher to question the relevance of the chord CD for the angle at the centre theorem. Early on in the discussion of D1, he stated:

> R: the thing I thought when I was looking at this first time around was, if the intention of this is to, for pupils to see that the angle at the centre is double that at the

circumference, then I wasn't necessarily sure about how actually drawing in the chord helped them with this. [Rob-GGb-int, 13.6.2012]

Later in the interview, Robert appeared to find a possible use for the chord CD in identifying the relevant angles in a particularly contorted configuration of D1 and, alluding to the angles in the same segment theorem, he also referred to the chord CD:

> R: I think dragging point B kind of demonstrates the, I mean, the invariance of the fact that well it's still coming from the same chord. [Rob-GGb-int, 13.6.2012]

In contrast, Edward incorrectly assumed that the chord CD *was* relevant to the angle in the centre theorem, using the chord in his attempt to define the angle at the circumference and the angle at the centre in a statement of the theorem:

> E: the chord C and D, joining C and D ... subtends an angle of 108 at the centre and 54 at the circumference, so uh … for this configuration of points it's showing that the angle at the centre is twice the angle at the circumference. [Ed-GGb-int, 20.6.2012]

Edward was the only case study teacher to volunteer an attempt at a precise definition of the angles involved in the angle at the centre theorem. The other three teachers left the definition of the angles implicit, generally referring to them as the 'angle at the circumference' and the 'angle at the centre'. Edward's attempted definition later caused him difficulties as he encountered the convex quadrilateral configuration of D1 where the relevant angles, i.e. those involved in the doubling relationship, did not conform to his definition based on the chord – see Figure 5.6, Edward's angle definition dilemma. Neither Anne nor Michael mentioned the chord CD.

Recognising the chord CD as extraneous seems an important example of foundational knowledge because of how the chord seemed to influence Edward and Robert's *choice and use of examples* and *decisions about sequencing*. In particular, perhaps as a result of defining the angle at the centre as subtended by the chord CD, Edward viewed the convex quadrilateral configuration shown in Figure 5.6 as a complication to be avoided rather than a natural continuation of the theorem. Similarly, Robert suggested he would drag B first, essentially demonstrating the angles in the same segment theorem, causing the convex quadrilateral configuration to first arise with the 'incorrect' angle at the centre. Thus, unfortunately, the convex quadrilateral configuration first appears as a non-standard or special case of the angle at the centre theorem due to the relative positioning of points B, C and D. The appearance of the convex quadrilateral configuration as a non-standard or special case may have been heightened by the presence of the chord CD, since it separates the circle into two segments, indicated by

Robert's reference to B being positioned in the "opposite segment" as one of two main special cases:

> R: I think I mean those are the two main ones I think in the opposite segment and also the DC being a diameter. [Rob-GGb-int, 13.6.2012]

The 'Arrowhead' Rule

Michael was the only case study teacher to state that "in my head this rule is the Arrowhead Rule" [Mic-GGb-int, 31.5.2012]. He said this in response to the interviewer's introduction of the upside-down arrowhead configuration, a 180-degree rotation of Figure 5.3 or Figure 5.8 (a) the standard arrowhead configuration, to indicate that the rotation did not seem a significant alteration to him. A little later, Michael moderated this response, after further prompting from the interviewer, to:

> Yeah, I guess the point is I mean not to call it the Arrowhead Rule so that they look for an arrowhead, because now they might look at that and go you know 'It's two spikes going up, I'm used to one spike going up' and then don't realise it's just upside down. So yeah, I guess yeah I guess there is a benefit to that. [Mic-GGb-int, 31.5.2012]

To some extent, Michael's moderated response is a positive instance of *identifying pupil errors* i.e. a recognition that pupils may find different orientations difficult to identify as the same configuration. On the other hand, his admission that, for him, the angle at the centre theorem is really the 'Arrowhead' Rule is a rather negative instance of *overt subject knowledge*, suggesting he views the arrowhead configuration as the standard configuration of the angle at the centre theorem, relegating other mathematically valid configurations of this theorem as non-standard, exceptional or special cases.

Although Michael was the only teacher to articulate that he saw the angle at the centre theorem as the Arrowhead Rule, the *choice and use of examples* and *decisions about sequencing* of the other case study teachers also suggested they might implicitly share the same understanding. Edward argued that he would drag points B, C and D, maintaining the arrowhead configuration, until he felt that his pupils had grasped the doubling relationship, only then moving on to "complicate" it by considering alternative configurations:

> E: What I'd start with is look just move B between C and D but don't cross it and move D just so it doesn't go further round than CD being a diameter. [Ed-GGb-int, 20.6.2012]

E: When I was happy everyone had got that idea for that sort of arrow-shape, I'd then move on and complicate it. [Ed-GGb-int, 20.6.2012]

Robert and Anne also suggested they would drag point B first, essentially demonstrating the angles in the same segment theorem, leaving the cross quadrilateral configuration implicit and causing the convex quadrilateral configuration to first arise with the 'incorrect' angle at the centre. As argued previously, this has the effect of causing the convex quadrilateral configuration to appear as a non-standard or special case, reifying the arrowhead configuration as the standard configuration.

Adhering to the starting configuration

Robert was the only case study teacher to consider modifying the starting configuration of D1 to suit his own pedagogical requirements. He suggested he might alter D1 so that the initial numerical example displayed when opening the GeoGebra file would be an almost implausibly 'nice' pair of numbers, setting the angle at the circumference to 60 degrees and the angle at the centre to 120 degrees as an example. His intention was to set up a situation that appeared 'too good to be true' so that pupils would assume no relationship was likely to exist and would therefore sustain cognitive conflict when the angle at the circumference remained invariant under drag, hopefully making the result more memorable. The other three case study teachers uncritically accepted the starting configuration, questioning neither the numerical example nor the geometric configuration.

The geometric nature of the starting configuration, in particular, is important since it provides an implicit pedagogic structuring. For example, opening the GeoGebra file so that D1 initially displays an arrowhead configuration implies a *choice and use of examples* and a *decision about sequencing* that alternative configurations will occur as a consequence of the arrowhead configuration, potentially reinforcing the impression of the arrowhead as the standard configuration of the angle at the centre theorem. An alternative would be to open the GeoGebra file so that D1 initially displays the convex quadrilateral configuration as a means of challenging this apparent orthodoxy. In addition, the starting configuration tends to impose *decisions about sequencing*, since some configurations are more difficult to obtain depending on whether they require dragging point B, C or D only or a combination of these points.

The case study teachers' adherence to or modification of the starting configuration appears to coincide with the meaning of the code *adherence to textbook*, in the sense that it describes a situation involving mathematical knowledge for teaching where a teacher decides either to adhere to or to modify the pedagogic structuring of mathematics by a teaching resource. The teacher's decision, implicit or explicit, regarding the pedagogic structure of the teaching resource provides an indicator of foundational knowledge. An implicit (i.e. uncritical) adherence to the pedagogic structure of the teaching resource implies a negative reading of the code. Thus Anne, Edward and Michael's uncritical acceptance of the starting configuration suggests they lack foundational knowledge that the starting configuration of D1 might be (usefully or otherwise) critiqued in terms of the pedagogic structuring it provides. Hence they make a *readerly* response to D1 (Bowe, Ball and Gold, 1992, drawing on the work of Barthes). Nevertheless, a readerly response might apparently result in a positive *choice and use of examples* say, if the pedagogic structuring of the resource was sound. Thus, confusingly, a readerly response could also be interpreted as a positive example of the code *adherence to textbook.*

A *writerly* response (Bowe et al., 1992) to D1 would entail a recognition that the starting configuration of D1 might be critiqued in terms of the pedagogic structuring it provides, resulting in an explicit decision either to adhere to or to modify the pedagogic structuring of the teaching resource. This suggests a positive reading of the code *adherence to textbook*. Indeed, an explicit decision to adhere to the pedagogic structuring of the teaching resource would be a positive example of the code *adherence to textbook* if the pedagogic structuring of the resource were sound. On the other hand, such a decision could also be interpreted as a negative example of the code if the pedagogic structuring turned out to be flawed in some way. An explicit decision to modify the pedagogic structuring of the teaching resource which resulted in improvement, would again indicate a positive example of *adherence to textbook* – this latter is also dealt with by the new code *use of instructional materials* under the Transformation category introduced by (Petrou & Goulding, 2011). However, Robert's decision to modify the starting configuration could be interpreted as a deterioration in the pedagogic quality of the initial choice of example: it is geometrically too close to being symmetric and the numbers are exceptional. Thus Robert's modification could be interpreted both as a positive and negative instance of the code *adherence to textbook* under Foundation and a negative example of *use of instructional materials*. This analysis is rather cumbersome and symptomatic of the Knowledge Quartet's relative lack of focus on knowledge in relation to teaching resources in general.

Finally, there is a slight discordancy in using this code to describe the case study teachers' adherence to or modification of the starting configuration, since the code specifically refers to a textbook and not a digital resource such as the GeoGebra file on circle theorems. The specificity of the adherence to textbook code derives from the non-digital technology context in which it was grounded. The discordancy may be ameliorated by a minor alteration to the code, so that it refers to a more generic teaching resource as in *use of instructional materials* (Transformation) or *responding to the (un)availability of tools and resources* (Contingency).

6.1.2 Transformation

Numerical examples

Numerical variation is crucial to appreciate the nature of the numerical relationship between the angle at the circumference and the angle at the centre as a doubling relationship as opposed to any other kind of numerical relationship, e.g. an additive one. It also indicates the range of numerical values over which the relationship is conjectured to hold. Providing numerical variation using D1 is essentially dependent on dragging either point C or D, since dragging point B leaves the angle at the centre invariant and only allows two possible numerical values for the angle at the circumference.

All the case study teachers, at some point in the interview, dragged points C and D and made an explicit *choice and use of examples* with regard to the numerical examples they thought worthwhile demonstrating to their pupils. Often these choices were articulated in terms of ensuring the doubling relationship would not be obscured by an onerous calculation or disguised by rounding errors, rather than an explicit consideration of the need to provide numerical variation for the purposes described above. For example, as described in Chapter 5, both Michael and Robert chose numerical examples where the angle measures were multiples of 10 to make the doubling calculation more straightforward, helping their pupils 'spot' the relationship. Anne and Edward expressed disquiet about the potential of numerical examples containing rounding errors to obscure the doubling relationship (see Chapter 5), and all four teachers acknowledged the need to draw pupils' attention to numerical examples containing rounding errors as a means of effacing the issue. For example, by dragging point D so that it coincided with point C, Michael found a particularly nice series of numerical examples with small angles as a means of addressing the issue of rounding errors, see Figure 6.2:

M: If I okay, so if I move C and D together, they're both going to be zero. Okay, that's good because as soon as you start moving C and D apart, so you get, yeah, that's again, that's rounding isn't it, because you've got the angle at the centre is 1. The angle at the circumference is zero, but then when you … that's a lovely illustration. It's gone from 2 to 1, but then you're going to get 3 to 1, so there's halves isn't there?

I: Right.

M: So 4 to 2, so you're going to get 5 to 2 before you get 6 to 3 or yeah. [Mic-GGb-int, 31.5.2012]

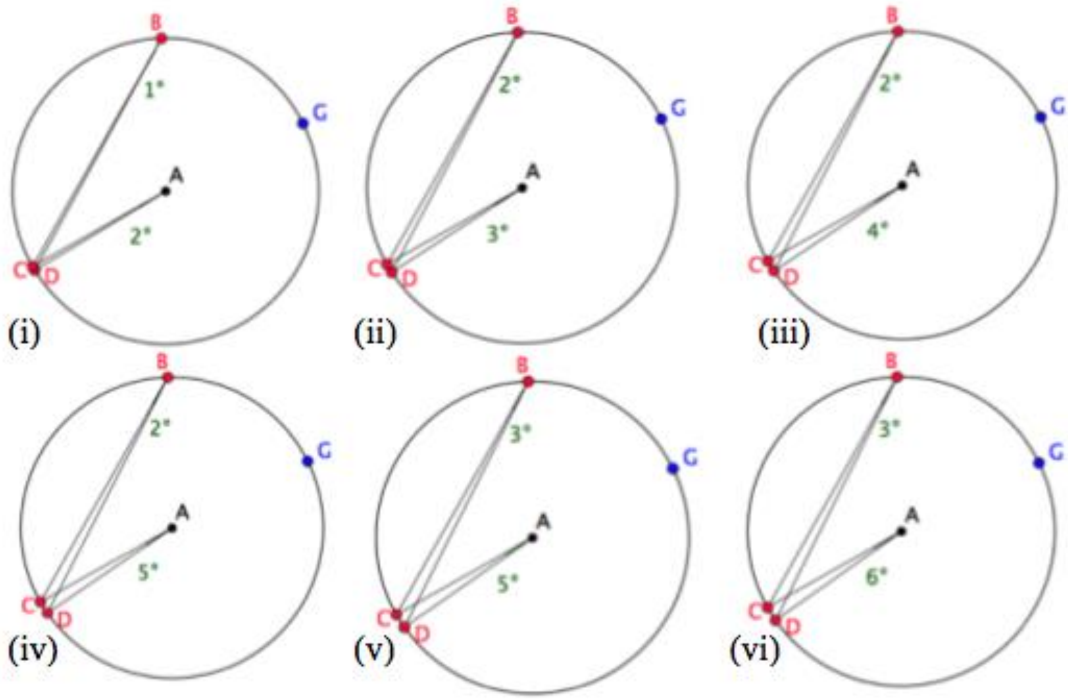


Figure 6.2 Michael's series of numerical examples with small angles. Note he does not mention (v) but it is included here for completeness.

Geometric configurations

Geometric variation is crucial to appreciate the sufficiency of the condition that points B, C and D lie on the circumference of the circle for the angle at the centre theorem to hold. That is, the theorem holds no matter where the three points are positioned on the circumference, so long as they are distinct from each other. The positioning of points B, C and D in relation to each other determines the 'shape' of the internal quadrilateral BDAC. Thus there are three standard geometric configurations of the angle at the centre theorem: the arrowhead configuration, the convex quadrilateral configuration and the cross-quadrilateral configuration (see Figure 6.3). All four case study teachers explicitly recognised the arrowhead and convex quadrilateral configurations and thought it important to demonstrate these to their pupils. For Robert, Anne and Edward, the cross-quadrilateral configuration first arose as a result of dragging point B around the major arc CD, when asked how they would demonstrate D1 for pupils. At this stage in their interviews, the cross-quadrilateral configuration remained implicit: none of the teachers articulated recognition of this configuration as a distinct example.

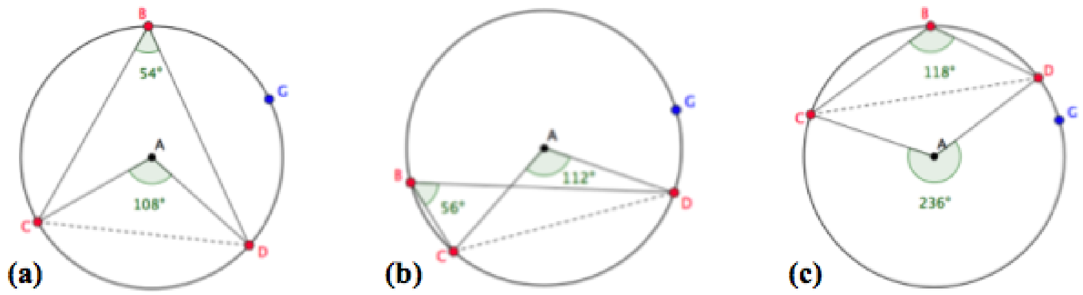


Figure 6.3 (a) The arrowhead configuration **(b)** the cross quadrilateral configuration and **(c)** the convex quadrilateral configuration

Later in their interview, each of these three teachers again came across the cross-quadrilateral configuration but as a result of dragging either point C or D. Robert, Anne and Edward each discussed this instance of the cross-quadrilateral configuration, explicitly recognising it as a distinct example. Their recognition of the cross-quadrilateral configuration in this instance was due to the ‘incorrect’ angle at the centre being displayed, requiring some thought as to whether the theorem still applied or had broken down. Michael first came across the cross-quadrilateral configuration in this manner, through dragging point D. Both Robert and Michael discussed whether this example was appropriate to show their pupils. Robert concluded that it might be useful for his students to struggle to verify the theorem still applied, in the manner he himself had done, though it remained unclear whether he would make a point of demonstrating it to them explicitly. Michael suggested he would probably demonstrate the example to his highest-attaining students, as long as he felt sure they had a strong grasp of the theorem already, but it seemed unlikely that he would demonstrate this example for slightly lower attaining students. The case study teachers’ decisions about whether a particular geometric configuration was suitable for demonstrating to their pupils provides another example of their *choice and use of examples.* It is worth noting here that a teacher’s recognition of a case (or not) is an example of Foundation knowledge. Having recognised a particular case as an example, a teacher’s choice about whether and how to use the case with pupils falls under the Transformation category.

Teacher demonstration

The code *teacher demonstration* provides an example of where the interview context appeared to disrupt the intended meaning of the code i.e. an instance of mathematical knowledge for teaching arising through a teacher demonstrating something mathematical to pupils in the context of a lesson. During the interview, there were no pupils present and, of course, to some extent, the case study teachers could be said to be demonstrating some aspect of circle theorems to the author at all times throughout the interview. However, the disruption of the teacher demonstration code appears more subtle than this statement would seem to imply.

At times during the interview, in particular when asked how they would demonstrate D1 for pupils, the case study teachers demonstrated to the author how they might drag the diagram for (imaginary) pupils. These instances remain as close to the original meaning of the code as possible in the interview context. At other times, the case study teachers appeared to demonstrate the diagram to the author as if to a colleague i.e. a fellow mathematics teacher from their department, for example. Robert provided the clearest instance of this in his initial description of D1, before being asked how he would use the diagram with pupils. He demonstrated his understanding of how the diagram is constructed; of the pedagogic purpose of the diagram i.e. to introduce the angle at the centre theorem and that there is an issue with rounding errors, which needs to be dealt with pedagogically.

> Okay, so we've got, so the three red points are presumably all fixed to the circumference of the circle and I can modify all of them so they're all free to move. Um, and what they do, so two of the points define a chord. The third point are connected to those two points and it … well, that chord, there's a subtended angle which is marked and measured and at the centre there's the same angle which in the way that it's currently looking is double [dragging B back and forth], but not always, if you choose numbers that round awkwardly. [Rob-GGb-int, 13.6.2012]

This demonstration of the diagram is quite different to how he later suggested he would demonstrate D1 to pupils. Following on from this demonstration, Robert continued to explore the diagram, this time dragging point C. Coming on an unexpected configuration, he seemed to be speaking to himself rather than to an observer: he appeared to speak more softly, asking questions that seemed directed to himself about what was happening, since he answered them himself.

> What happens if it comes this way? [dragging point C, stops at angles B=100, A=200] That still works. And if that comes over there? [dragging C past B, ‘incorrect’ angle at centre shown] It doesn’t work anymore. [Rob-GGb-int, 13.6.2012]

The other case study teachers similarly appeared to ‘demonstrate’ the diagram to themselves when confronted with unexpected configurations. Thus, in an interview context, there appears to be three types of demonstration: demonstration to (imaginary) pupils; demonstration to a colleague/the interviewer, and ‘demonstration’ to oneself i.e. personal exploration. The first two types of demonstration belong under Transformation, since they suggest a demonstration of the diagram for pedagogic purposes, whilst the third is more appropriately considered under Foundation as it indicates a personal understanding of mathematics.

6.1.3 Connection

Sequencing configurations

All the case study teachers made *decisions about sequencing* in terms of the order in which they introduced geometric configurations to their pupils. For example, Edward suggested he would maintain the arrowhead configuration, first dragging point B and then points C and D, until he felt his pupils had a good grasp of this configuration, before introducing new configurations as a “complication” [Ed-GGb-int, 20.6.2012]. As a first complication, Edward suggested he would introduce the convex quadrilateral configuration by dragging point B onto the minor arc CD. Secondly, Edward suggested he would drag point D so that the chord CD becomes a diameter. Finally, he suggested he might continue to drag point D, thereby introducing a different version of the convex quadrilateral configuration. Edward’s sequencing of geometric configurations is depicted in Figure 6.4.

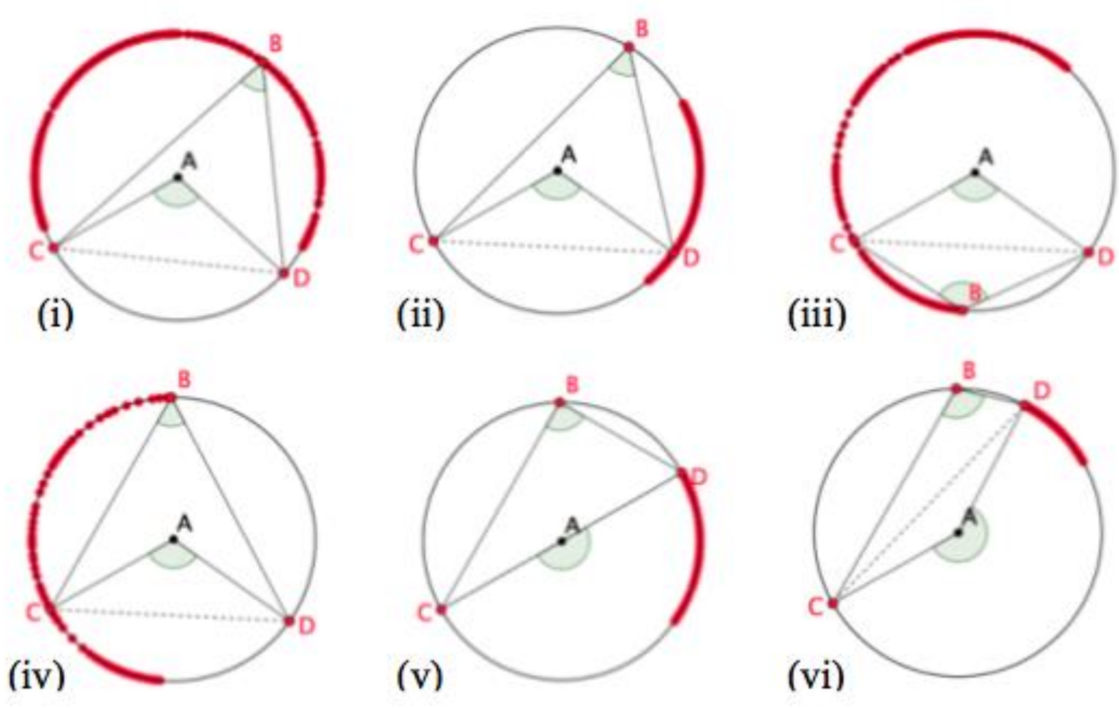


Figure 6.4 An indication of Edward's sequence of configurations. The trace gives a sense of how he dragged points B, C and D.

Similarly, Robert and Anne suggested they would maintain the arrowhead configuration initially, before introducing the convex quadrilateral configuration by dragging point B onto the minor arc CD. In addition, Robert later suggested he would also introduce the configuration where CD is a diameter and the alternative version of the convex quadrilateral configuration arrived at through dragging point D.

Michael also suggested he would maintain the arrowhead configuration initially whilst dragging points B, C and D. However, in contrast, he then suggested he would introduce the convex quadrilateral configuration by dragging points C and D, as shown in Figure 6.5. Michael's sequencing produces a different impression of the convex quadrilateral configuration and is preferable to the other three case study teachers' *decisions about sequencing* for reasons discussed in Chapter 5.

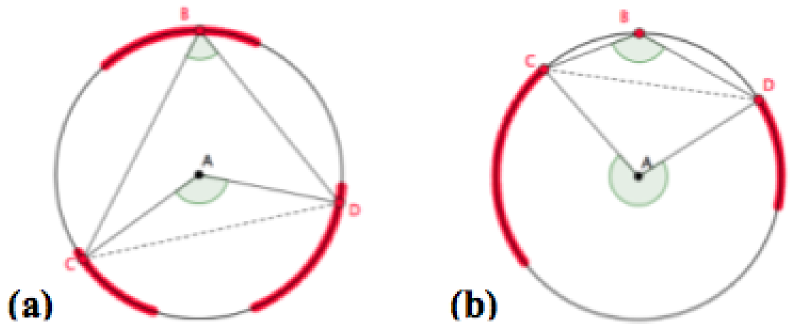


Figure 6.5 An indication of Michael’s sequence of configurations

Making connections between circle theorems

Using D1 to make connections between circle theorems is an example of *making connections between concepts*. Robert wanted to use D1 to make connections between the angle in the centre theorem and two other circle theorems, the angles in the same segment theorem and ‘Thales theorem’ or the angle in a semi-circle is right. Similarly Edward thought that dragging point D so that CD forms a diameter made a “nice” connection, showing “really well” that the angle in a semi-circle is right is a consequence of the angle in the centre theorem [Ed-GGb-int, 20.6.2012].

Making these connections between the circle theorems impacted on the case study teachers’ *choice and use of examples* and *decisions about sequencing*. In particular, Robert and Edward both dragged point B to initiate their demonstration of D1 to pupils, which is in essence a demonstration of the angles in the same segment theorem (see Figure 6.4 i and iii). Whilst dragging point B does make a connection between the two theorems and may connect with prior work, as Robert suggested, the *decision about sequencing*, i.e. choosing to introduce D1 in this way, risks distracting from the main pedagogic purpose of diagram, namely to introduce the angle in the centre theorem. Pedagogically, it seems preferable to use D1 to make connections with the angle in the same segment theorem once the angle in the centre theorem has been established. The angle in the semi-circle theorem by contrast can be viewed as a particular numerical example of the angle in the centre theorem. Thus choosing to make this connection between the two theorems represents a positive case of the *choice and use of examples*. In addition, making this connection by dragging point C and/or point D represents a positive case of *decisions about sequencing*, leading onto the introduction of the convex quadrilateral configuration, since providing both geometrical and numerical variation is crucial to demonstrating the angle at the centre theorem, as argued above. Figure 6.6 provides an illustration of what such a dragging sequence might look like.

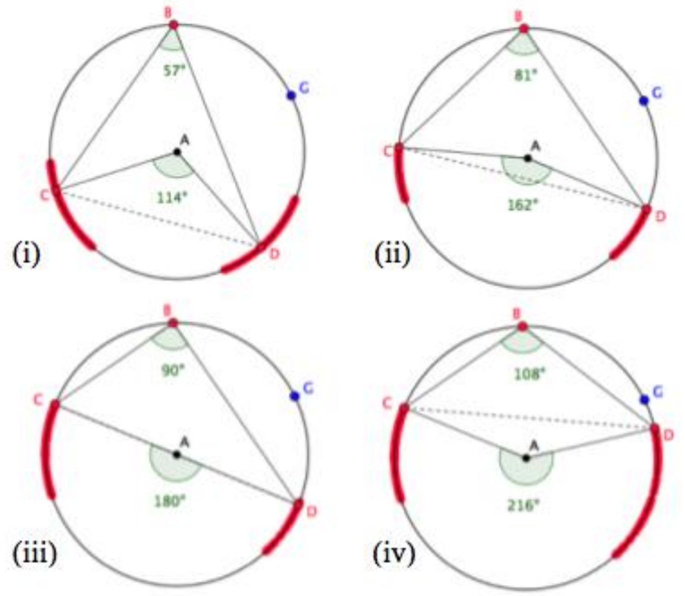


Figure 6.6 Illustration of an introductory dragging sequence for D1, connecting the angle in the centre theorem to the angle in a semi-circle is right theorem

6.1.4 Contingency

Unexpected configurations

During their dragging of the diagram, each of the case study teachers came across unexpected configurations of D1 that gave rise to *teacher insight*. Typically, these configurations were unexpected either due to their unusual orientation or because the 'incorrect' angle was displayed at the centre, thus the doubling relationship was not readily perceptible (see Figure 6.7 for examples). The case study teachers had to put some effort into reconciling these unexpected configurations with their existing knowledge, to recognise the angle at the centre theorem still held, entailing a (perhaps unstable) change in their Foundation *overt subject knowledge*. Here, a critical part of overt subject knowledge appears to be the range of examples of the angle at the centre theorem the teacher has explicit knowledge of in relation to D1 and in the context of the interview, i.e. their evoked concept image (Tall & Vinner, 1981). It is this set of explicitly known examples, evoked both in and by the technological and interview context, that the case study teachers can draw on in a transformation of their knowledge for pedagogic purposes through a deliberate *choice and use of examples.*

In addition, at times prompted by the author, the case study teachers sometimes considered whether and how they would use such a configuration if it arose during the course of a lesson. The convex quadrilateral configuration that arose during Robert's interview by dragging point B onto the minor arc CD, described in the section on sequencing configurations, is an example of such an instance. Robert concluded that if he had known this configuration was going to arise in the lesson, he might deliberately avoid it, at least initially, in the hope that one of his pupils might raise it instead. He stated that previously he had constructed diagrams so that B was forced to lie on the major arc CD, so that such a configuration could not occur, but did not think he would do this in future. In the context of the interview, Robert's conclusion represents a *deviation from lesson agenda* since it involves a change in his knowledge of how to transform the diagram for pedagogic purposes, in particular, a change in his *choice and use of examples*.

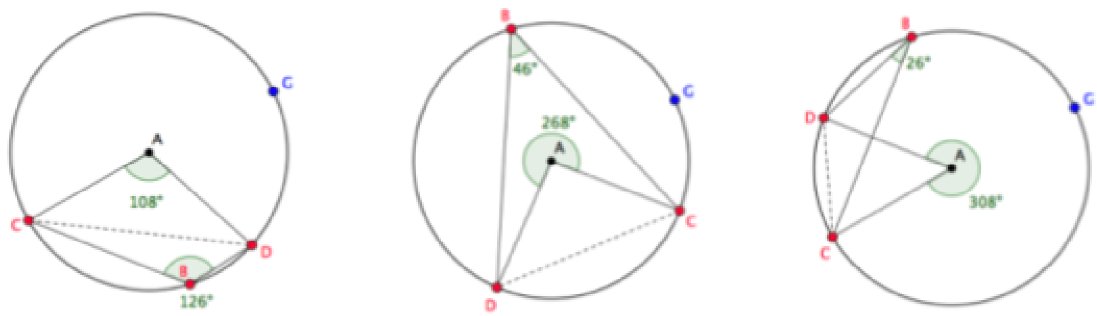


Figure 6.7 Three examples of case study teachers' unexpected configurations

Two absences and a presence

The codes *responding to student's ideas* and *responding to the (un)availability of tools and resources* from Contingency were necessarily absent from the data. Given the interview design, based around a GeoGebra file, there were necessarily no tools and resources either unexpectedly present or unavailable. Without the necessary technology, the interview could not take place and there was no remit within the interview protocol for introducing additional tools or resources. Trivially, there were no children present during the interview, to whom the case study teachers could respond.

One type of contingency was not covered by the Knowledge Quartet codes: when the author introduced configurations as a prompt for the case study teacher, as in the upside-down arrowhead configuration, for example. These prompted configurations were generally unexpected, at least for the case study teacher. It is conceivable that an analogous contingency might occur in the context of a lesson where a teaching colleague was present, for example, due to team-teaching or due to the presence of a teaching assistant – the latter a reasonably common situation in current UK classrooms. Such a code might be termed *responding to colleague's intervention*. The contingencies caused by the author's prompts generally revolved around *choice and use of examples*, for example, of what pedagogic benefit was the upside-down arrowhead i.e. was it worth demonstrating to pupils. For Edward, this configuration also raised *decisions about sequencing*: he was concerned about the dragging sequence required to effect this configuration.

6.1.5 Summarising the Knowledge Quartet analysis

The analysis above demonstrates the emergence of *choice and use of examples* and *decisions about sequencing* as central codes in describing mathematical knowledge for teaching circle theorems using technology. The centrality of *choice and use of examples* in describing mathematical knowledge for teaching circle theorems using technology coincides with research emphasising the importance of variation for learning and pedagogy in general and for mathematics in particular (Marton & Booth, 1997; Watson & Mason, 2005; 2006). The centrality of *decisions about sequencing* from the Connection category of the Knowledge Quartet provides evidence in support of the hypothesis, stated in the introduction to this chapter, that mathematical knowledge for teaching using technology might be measured in terms of the connectedness of teachers' mathematical knowledge. In particular, whilst variation in terms of choosing a set of examples is pedagogically important, the sequencing of these examples to provide a structure, in which attention is drawn to particular invariants before others, is equally important for pedagogic purposes. For example, choosing to introduce D1 by dragging point B produces variation which draws attention to the invariance of angles in the same segment theorem, thus distracting from the pedagogic purpose of the diagram, namely, to introduce the angle at the centre theorem. Introducing D1 by dragging B first had the added disadvantage of causing the convex quadrilateral configuration to seem like a "complication" rather than a natural consequence of the angle at the centre theorem. In other words, providing numerical variation to make connections between different configurations of the same theorem *before* making connections with other theorems seems to be preferable for pedagogic purposes, indicating mathematical knowledge for teaching circle theorems using technology. Thus any measure of mathematical knowledge for teaching circle theorems using technology should reflect both the choice of examples and their sequencing.

In addition, ideally a measure of mathematical knowledge for teaching circle theorems using technology should reflect the distinction, made in the section entitled '*Unexpected configurations'* under Contingency, between examples (i.e. geometric configurations) that are explicitly recognised by the teacher and those that arise during the teacher's or the author's manipulation of the diagram but which remain implicit. This is important because it may be the mathematical knowledge made available in the classroom i.e. the knowledge distributed across both teacher and technology, rather than the individual teacher's own knowledge that has most impact on the quality of pupils' mathematical experience in the classroom. In making a deliberate *choice and use of examples*, the case study teachers were limited to the set of their explicitly known examples, evoked both in and by the technological and interview context i.e. their evoked concept image (Tall & Vinner, 1981). A further distinction was made between those explicitly-recognised configurations that the teacher chose to include through a deliberate *choice and use of examples*, and those they chose to exclude, either by suggesting they might discuss such configurations (only) if they happened to arise through dragging or by suggesting they might deliberately avoid them when demonstrating D1 to pupils. Again, a measure of mathematical knowledge for teaching circle theorems using technology should ideally reflect this distinction.

The analysis presented in the previous section shows that the Knowledge Quartet remained a useful tool for focusing analysis on mathematical knowledge despite the shift away from the classroom context in which the framework was originally developed and grounded. This finding is not entirely surprising since although the classroom is a particularly important context, it is not the only context in which teachers are likely to employ their mathematical knowledge for teaching. Nevertheless, the change in context, both in terms of the interview setting and the technological context, did result in two absences, notably under Contingency: a lack of situations requiring a response to pupils' ideas and a lack of situations requiring a response to the (un)availability of tools and resources. This reflects the findings of Chapter 5 that mathematical knowledge for teaching using technology is a situated abstraction, simultaneously representing 'abstract' mathematical knowledge that can be applied across contexts and yet situated knowledge in that the particular context in which it was developed may be central to its meaning.

In addition, the analysis of the GeoGebra interview data suggested some minor modifications to the Knowledge Quartet, both in relation to technology and to the interview context, that might prove useful when re-applied back to the original classroom setting or to other settings where teachers employ their mathematical knowledge for teaching. For example, *responding to colleague's intervention* was suggested as an additional code under Contingency that might be applicable beyond the interview context. A distinction was also made between three types of teacher demonstration, i.e. a demonstration to pupils versus a colleague under Transformation and to oneself under Foundation. This distinction seems less applicable beyond the interview context, since it is hard to envisage how the latter two types might occur in a classroom context.

The cumbersome analysis of situations involving the code *adherence to textbook* is symptomatic of the Knowledge Quartet's lack of focus on knowledge in relation to (digital) technology. The difficulty with this code is that it categorises situations involving the application of foundational knowledge both in perceiving the technology as something requiring a pedagogic critique and in terms of the quality of the critique applied to the teaching resource. The former relates to the teachers' foundational knowledge in adopting a *readerly* or *writerly* approach to the resource (Bowe et al., 1992). The latter is also dealt with under the Transformation category, specifically the code *use of instructional materials*, which additionally reflects back onto the teachers' foundational knowledge indicated by the quality of the pedagogic critique applied to transform the resource for the purpose of teaching. There is no easy way to ameliorate this difficulty within the Knowledge Quartet, however, *adherence to textbook* could be modified to reflect a broader range of teaching resources rather than privileging this paper-based technology. Furthermore, as a result of new codes added by a range of researchers, the codes of the Knowledge Quartet use an impromptu variety of terms to refer to teaching resources including textbook, instructional materials, tools and resources. The variety of terms is not intended to make any productive distinctions, as far as the author is aware, thus it might simplify and improve the coherence of the

Knowledge Quartet to settle on a particular term or group of terms to refer to teaching technologies. Finally, the analysis above provides an exemplification of the Knowledge Quartet in relation to digital technologies, albeit not in a classroom context. This exemplification might be useful in helping teachers to use the Knowledge Quartet as a tool for professional development in relation to their use of technology, as described in Turner and Rowland (2011).

The next section builds on these findings to justify the development of the *circle theorem case list* as a potential tool for measuring mathematical knowledge for teaching circle theorems using technology.

6.2 Developing a tool for measurement: the circle theorem case list

A *circle theorem case list* is a numbered sequence of the pedagogic cases of the angle at the centre theorem in the order they arose through an individual case study teacher's dragging of D1 during their semi-structured GeoGebra interview. Describing the development of this measurement tool shows how this process was necessarily entwined in a dialectic relationship with the conceptualisation of mathematical knowledge for teaching circle theorems using technology (Hill et al., 2008). In particular, it shows how the process of developing the circle theorem case list was crucial to unravelling the complexity of this knowledge. However, due to the complexity of mathematical knowledge for teaching circle theorems using technology, it seems expedient first to describe in detail the final measurement tool to provide a context for understanding the process of tool development. Thus the next sub-section provides a detailed description of the circle theorem case list, showing how the measurement tool allows comparisons to be made between the case study teachers' *choice and use of examples* and *decisions about sequencing*. In addition, the analysis of interview data above using the Knowledge Quartet is employed to provide a post-hoc justification that the circle theorem case list shows potential as a measurement tool. This section then returns to provide a description of the process of developing the circle theorem case list. Finally, section 6.3 considers how the conceptualisation of mathematical knowledge for teaching using technology in relation to the topic of circle theorems, presented here, suggests ways in which such knowledge could be measured. The chapter concludes by speculating tentatively about the relationship (if any) between a connectionist versus transmissionist teaching orientation and mathematical knowledge for teaching circle theorems using technology.

6.2.1 A description and justification of the circle theorem case list

The circle theorem case list provides a means of summarising the case study teachers' *choice and use of examples* and *decisions about sequencing* in a manner that facilitates comparison. This fulfils the main criterion for a potential tool for measuring mathematical knowledge for teaching circle theorems using technology identified above in the analysis of the interview data using the Knowledge Quartet. Figure 6.8 presents the circle theorem case list for each of the four case study teachers. The reader is advised to view Figure 6.8 alongside Figure 6.9. To this end, both figures are included as separate sheets in the back cover of the thesis. The numbers in Figure 6.8 indicate the pedagogic cases of the angle at the centre theorem, elicited during the case study teacher's interview. The numerical list of these pedagogic cases is depicted in Figure 6.9 in order of the ease with which they can be obtained from the starting configuration through dragging points B, C and D. The pedagogic cases are numbered in this way in an attempt to preserve the case study teachers' patterns of dragging, in effect the pedagogic structuring of cases imposed by the construction and starting configuration of D1. The list of pedagogic cases is also represented as a table in Figure 6.10 to demonstrate that the list represents a complete set of cases, given the criteria under which they were identified. The table was constructed so that its columns indicate the set of pedagogic cases that might be identified in an environment where dynamic variation through dragging the diagram was not possible, such as a textbook. That is, the columns indicate a set of eight standard, special and extreme geometric configurations, together with the recognition that for configurations where the angle at the centre is not a multiple of 180, displaying a reflex angle at the centre constitutes a separate pedagogic case to the configuration where an angle less than 180 degrees is displayed. The recognition of configurations displaying the reflex and non-reflex angles at the centre as separate pedagogic cases is important due to the relative difficulty of apprehending the doubling relationship at the heart of the angle at the centre theorem when the 'incorrect' angle at the centre is displayed. The table was constructed with two rows to reflect the dynamic nature of the GeoGebra software. Thus, a configuration was considered to constitute a separate pedagogic case if it arose as a result of dragging point B or as a result of dragging points C and/or D. This distinction is important since dragging points C and D results in numerical variation, providing evidence for the doubling relationship between the angle at the centre and circumference, whereas dragging point B only varies whether the 'correct' or 'incorrect' angle at the centre is shown, as a consequence of the angles in the same segment theorem.

The numbers representing pedagogic cases in each of the circle theorem case lists are also formatted, using **bold**, *italics*, underline, and grayscale, according to the following coding system, to indicate whether the pedagogic case arising during the interview was:

(a) chosen by the case study teacher to show pupils in a deliberate *choice and use of examples*;

(b) explicitly recognised by the case study teacher or remained implicit; and

(c) unprompted, prompted indirectly or prompted directly by the author.

The specification of the coding system is described in Table 6.1 and the details of how formatting is used to indicate this coding are provided as part of Figure 6.8. This coding system fulfils the secondary criteria for a potential tool for measuring mathematical knowledge for teaching circle theorems using technology that such a tool should reflect the distinctions made in the analysis of the interview data using the Knowledge Quartet, in the section entitled '*Unexpected configurations'* under Contingency. These distinctions were two-fold, involving (1) differentiating between the pedagogic cases which appear to form part of the case study teacher's evoked concept image, and those that arise during the teacher's or the author's manipulation of the diagram but which remain implicit and (2) distinguishing between those cases forming part of the case study teacher's evoked concept image that the teacher chose to include through a deliberate *choice and use of examples*, and those they chose to exclude for whatever reason.

Table 6.1 Specification of coding system for pedagogic cases

Show Pupils	Case recognition	Prompting
Did the case study teacher state they would show their pupils the pedagogic case in a deliberate *choice and use of examples*? **Yes** The case study teacher stated they would show this pedagogic case to their pupils. **No** Unclear whether the case study teacher would have shown this pedagogic case to their pupils e.g. stated they might avoid the case (initially); case would be acceptable if it comes up through exploration; case remained tacit.	Did the case study teacher recognise the pedagogic case explicitly or did it remain implicit? **Explicit recognition** The case study teacher *commented verbally* on the pedagogic case, whilst D1 was held stationary or left static showing the case. **Implicit recognition** D1 was held stationary or left static, if only briefly, but the case study teacher did not comment verbally on the pedagogic case OR the pedagogic case arose repeatedly during continuous dragging to show a 'family' of cases but was not referred to e.g. for Robert, Anne and Edward the cross-quadrilateral configuration arising from dragging point B initially. **Not Listed** Pedagogic cases were not included in the case study teacher's circle theorem case list if they did not arise during the interview or only occurred briefly during continuous dragging i.e. D1 was not stationary or static; continuous dragging was not to show a 'family' of cases.	How was the case study teacher prompted by the author to recognise the case? **Unprompted** Pedagogic cases that arose from the case study teacher's dragging after the first two interview questions: 'what's going on in this diagram?' and 'how would you drag it for pupils?' **Indirectly prompted** Pedagogic cases that arose following subsequent interview questions from the author, but were not directly introduced by the author. **Directly prompted** Pedagogic cases that were introduced directly through dragging by the author.

Robert*	**1**	***3***	**5**	**6**	13	4	12	10	***21***	***17***	***16***	***19***	*8*	*12'*	*6'*	1'	*3'*		
Anne	**1**	***3***	**5**	**4**	6	*8*	10	*10'*	6'	*12'*	***17'***	16'	21						
Edward	**1**	***3***	**5**	**6**	12	13	*10'*	6'	***12'***	19	*8*	10	*7*						
Michael	**1**	**6**	**12**	9	*8*	11	10	3	5	***7'***	***10'***	19	16	23	*7*	*21*	6'	1'	

Key to formatting of pedagogic cases

Show Pupils	Case Recognition	Prompting
Yes - underline No - normal text	Explicit - normal text Implicit - *italics*	Unprompted - **bold** Indirect prompt - normal text Direct prompt - grayscale

6 = yes, would show pupils; explicit case recognition; unprompted

6 = no, would not show to pupils; explicit case recognition; indirectly prompted

6 = no, would not show to pupils; explicit case recognition; directly prompted

6 = no, would not show to pupils; implicit case recognition; indirectly prompted

6' = indicates a 180-degree rotation of pedagogic case 6

Cases are listed in the order they arose in the interview with the exception of Robert*. If a case arose repeatedly, only the first instance is shown and the formatting shows the strongest code for the *case recognition* and *show pupils* variables across these instances. For the *prompting* variable, the formatting shows the coding of the first instance the case arose in the interview.

Robert* - presents the pedagogic cases as they arose through Robert's dragging following the second question 'how would you drag it for pupils?' This seemed to provide a fairer comparison with the other three case study teachers. The other three case study teachers suggested their dragging in response to the first question was indicative of how they would drag D1 for pupils, whereas Robert's dragging was significantly different.

Figure 6.8 Circle theorem case lists

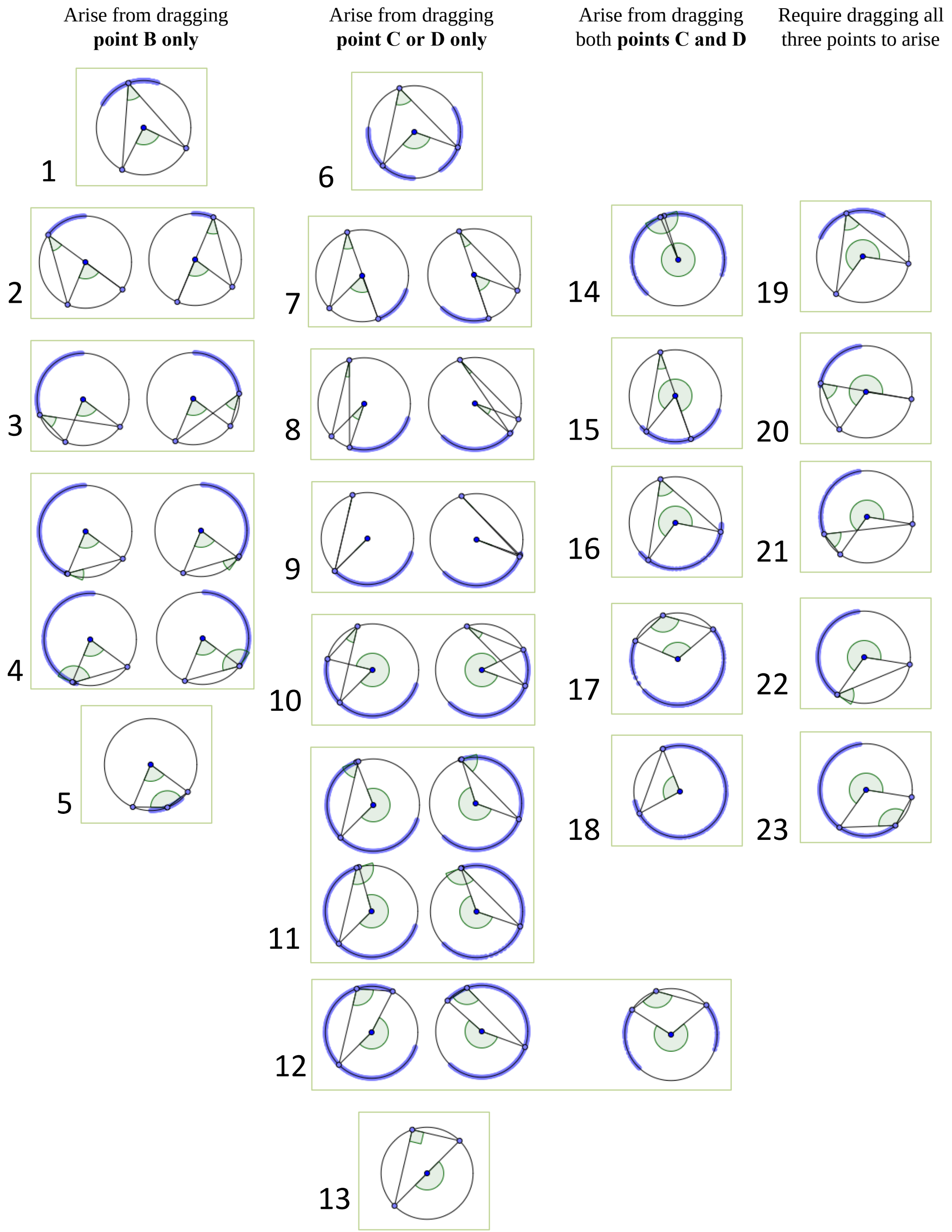


Figure 6.9 Numerical list of pedagogic cases

<table>
<tr><td></td><td colspan="6">‘Standard’ configurations</td><td colspan="3">Special configurations</td><td colspan="4">Extreme configurations</td></tr>
<tr><td>Configuration</td><td colspan="2">AH</td><td colspan="2">XQ</td><td colspan="2">CQ</td><td colspan="2">One-leg AH</td><td>Semi-circle</td><td colspan="2">Odd triangle</td><td>V-shape</td><td>T-shape</td></tr>
<tr><td>Angle at centre</td><td>NR</td><td>R</td><td>NR</td><td>R</td><td>NR</td><td>R</td><td>NR</td><td>R</td><td>180</td><td>NR</td><td>R</td><td>0</td><td>360</td></tr>
<tr><td>B</td><td>1</td><td>19</td><td>3</td><td>21</td><td>5</td><td>23</td><td>2</td><td>20</td><td rowspan="2">13</td><td>4</td><td>22</td><td rowspan="2">9</td><td rowspan="2">14</td></tr>
<tr><td>CD</td><td>6</td><td>16</td><td>8</td><td>10</td><td>17</td><td>12</td><td>7</td><td>15</td><td>18</td><td>11</td></tr>
</table>

NR = angle at the centre is between 0 and 180 degrees i.e. not reflex

R = angle at the centre is between 180 and 360 degrees i.e. reflex

B and CD indicate the last point dragged to arrive at the configuration.

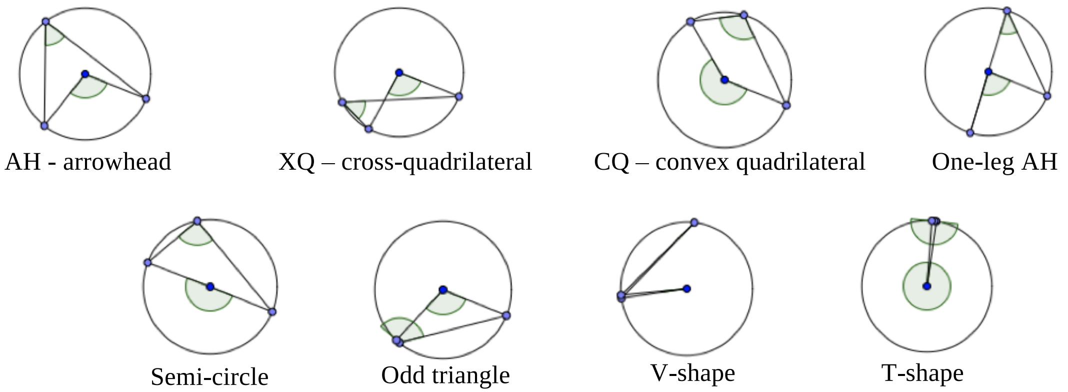


Figure 6.10 Tabular list of pedagogic cases

6.2.2 Developing the circle theorem case list: the coding system and list of pedagogic cases

Initially, the idea for a circle theorem case list appeared to originate spontaneously from the data, perhaps as a welcome by-product of coding the data using the TPACK and Knowledge Quartet frameworks. Following the presentation of the case study teachers' final circle theorem case lists in the last section, this spontaneity seems hard to credit since the analysis of the interview data using the Knowledge Quartet provides a post-hoc justification of the features of the circle theorem case list as a potential tool for measuring mathematical knowledge for teaching circle theorems using technology.

The circle theorem case list began as a rough attempt to create a numerical list of the configurations encountered by each case study teacher through their dragging of D1. The initial attempt was worth developing further because it seemed to provide a means of depicting how Michael's dragging sequence differed from the other three case study teachers (for example, see Figure 6.8). From this initial attempt, a working list of possible pedagogic cases arising from D1 was compiled. At this early stage of development, this working list comprised the standard arrowhead, cross-quadrilateral and convex quadrilateral configurations, recognising that displaying the incorrect angle at the centre might constitute a separate case for pedagogic purposes, as well as a selection of special and extreme cases. Due to the use of the upside-down arrowhead configuration as an interview prompt, 180-degree rotations were also included in the working list as separate cases in an ad hoc fashion, as and when they occurred through the case study teachers' dragging.

A second stage of development attempted to make more rigorous the method of identifying pedagogic cases, by specifying the conditions that would have to be fulfilled for inclusion in a case study teacher's circle theorem case list. The coding system detailed in Table 6.1 and described briefly in the previous sub-section resulted from this second developmental stage. A decision was made that a pedagogic case would be included if D1 was left static or held stationary so that it was possible for the author to recognise the configuration from the video in real time. In addition, a pedagogic case was included if it arose repeatedly as the result of continuous dragging as if to show a 'family' of cases – the cross-quadrilateral configuration caused by dragging point B initially is the main example of such inclusion. A narrative of the GeoGebra interview (see methodology, Chapter 3) was used alongside the video to guide observation.

Guiding observation using the narrative increased the efficiency of compiling a list of pedagogic cases arising in the interviews, by identifying static periods (where no new cases arose), which could be skipped over. As the author watched each video, the pedagogic cases were noted on the narrative as they arose and compared to the description of events provided by the narrative. Noting pedagogic cases on the narrative located them within the interview and also highlighted the location of author prompts in relation to where pedagogic cases arose. This location was useful for comparing interviews to verify the reliability of coding criteria across case study teachers. In addition, the process of comparing the narrative against a re-viewing of the video provided a form of triangulation over time as evidence of reliability.

Using the narrative to guide observation helped to identify a clear rationale for discriminating between explicitly and implicitly recognised pedagogic cases based on verbal data, resulting in the specification of the 'Case Recognition' variable detailed in Table 6.1. The original purpose of developing the narratives had been to provide an accessible summary of each interview, linking verbal and visual data together. Thus the pedagogic cases that occurred in the narrative generally did so because they were cases that the case study teacher had cause to comment on verbally, implying explicit recognition. Pedagogic cases that arose but did not feature in the narrative were those that did not excite verbal comment. Without verbal evidence, there was no support for a claim that the pedagogic case had been explicitly recognised by the case study teacher. Hence these pedagogic cases were included in the case study teacher's circle theorem case list but were coded as remaining tacit or implicit.

Coding the upside-down arrowhead configuration, in particular, led to a need to distinguish further between pedagogic cases according to whether they were introduced by the author or arose through the case study teacher's dragging, leading to the specification of the 'Prompt' variable detailed in Table 6.1. The upside-down arrowhead was introduced by the author as a direct prompt in each of the four GeoGebra interviews. Thus each case study teacher necessarily commented in response, as recorded in the narrative of their interview, and the upside-down arrowhead was coded as being an explicitly recognised pedagogic case. However, it seemed misleading to code the upside-down arrowhead in the same way as pedagogic cases that arose without direct intervention from the author. Further, the dragging that gave rise to some of the pedagogic cases, recognised by the teachers either implicitly or explicitly, appeared to be triggered in response to a prompt by the author. For example, Michael seemed to have finished with D1 until the author asked if there were any special or extreme cases he would want to show pupils. He responded by saying “There is now you’ve said that” and proceeded to drag D1 such that points C and D coincided to produce the ‘V-shape’ configuration. Thus it seemed reasonable to distinguish between pedagogic cases that were largely unprompted, arising as a result of dragging in response to the first two questions (“what’s going on in this diagram?” and “how would you drag it for pupils?”); those that were indirectly prompted through follow-up questions, as for Michael’s V-shape configuration; and those that were introduced as direct prompt by the author e.g. the upside-down arrowhead configuration.

Finally, in the second stage of development, the case study teachers appeared to place greater pedagogic value on some configurations than others, stating explicitly they would make a point of showing some configurations to their pupils, whereas for other configurations they remained ambivalent or even suggested they might avoid them. This led to the specification of the ‘Show Pupils’ variable detailed in Table 6.1. For example, Anne stated she would show her pupils the upside-down arrowhead configuration as well as the standard arrowhead and convex quadrilateral configurations. She was ambivalent to other pedagogic cases she came across, neither suggesting she would deliberately show or avoid them. Robert also stated he would show his pupils the upside-down arrowhead configuration, however Michael was largely ambivalent, whereas Edward was concerned by the dragging procedure used to obtain the configuration. Again the narrative was useful for helping to identify and locate pedagogic cases in the interview where the teacher stated they would show the configuration to pupils, since these pedagogic cases clearly gave cause for the teacher to make a verbal comment.

A third stage of development was prompted by the need to ensure that all pedagogic cases arising through the case study teachers’ dragging had been included in their case lists. Until this stage, if a pedagogic case arose repeatedly in interview, the circle theorem case list had only recorded the first instance of the case. Producing a case list including repetitions or multiple instances of pedagogic cases provided an opportunity to check that all pedagogic cases had been included. Thus a *visual transcript* of each interview was created to aid the production of a case list including multiple instances and as a means of ensuring that that all pedagogic cases arising through the case study teachers’ dragging had been included. A visual transcript is a pictorial list of configurations made visible by the case study teacher through dragging D1, whether recognised tacitly or explicitly or not at all (see Appendix H). Thus a visual transcript includes configurations of D1 that were momentarily visible during continuous dragging as well as those where the diagram was briefly stationary or static. To some extent, the visual transcript also recorded which point was being dragged to elicit a particular configuration although this was not done consistently. The production of a visual transcript entailed viewing the video of the interview at different speeds. For example, a short sequence of the video was viewed to 'get a feel' for what had transpired, then the video was viewed in a stop/start manner to ensure all cases were recorded. The video recording was sufficient to look at several configurations per second in this manner. At times, the dragging was too rapid for the author to observe all the intermediate cases when the video was played at normal speed and very difficult to isolate the cases by using the stop/start method, thus although the diagram must logically have passed through these cases, they were not recorded. The video of each interview was then viewed once again at normal speed for the full length of the discussion of D1, without stop/starting, to verify the visual transcript was accurate.

The creation of the visual transcript indicated that the working list of pedagogic cases was inadequate since it did not represent a complete, systematic list of pedagogic cases. In particular, it forced a deliberate consideration of visually similar pedagogic cases arising through different dragging approaches. For example, the two cases in Figure 6.11 are visually similar in that they are both cross-quadrilateral configurations when left static or held stationary, however dynamically they appear quite different since one arises through dragging point B and the other from dragging point C or D. The creation of the visual transcript prompted the recognition of the two configurations in Figure 6.11 as dynamically distinct pedagogic cases.

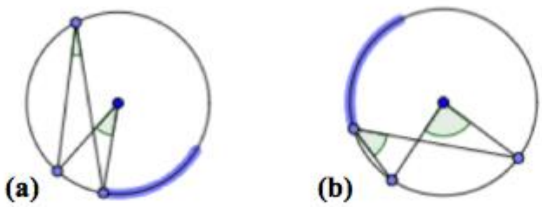


Figure 6.11 Dynamically distinct pedagogic cases of the cross-quadrilateral configuration arising through dragging **(a)** point D and **(b)** point B.

Recognising that pedagogic cases could be dynamically distinct entailed some deliberate decisions about what would constitute a distinct pedagogic case to prevent the number of cases becoming unmanageable. In particular, a decision was made that configurations arising through the clockwise or anti-clockwise dragging of a point are considered the same and similarly reflections in a vertical axis are considered the same. Thus the four configurations in Figure 6.12 are considered to be examples of the same pedagogic case.

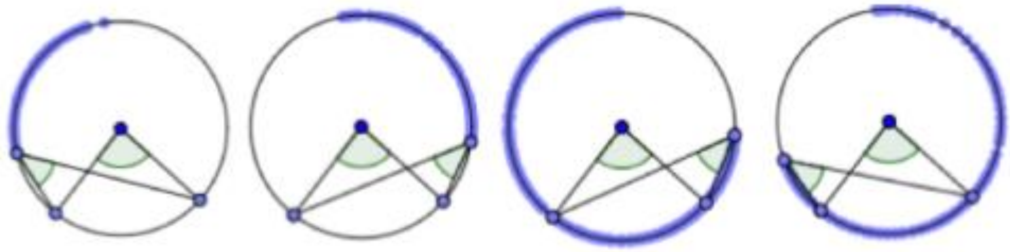


Figure 6.12 Four examples of the cross-quadrilateral configuration, all representing the same pedagogic case.

Points C and D are symmetrical so configurations arising from dragging point C were considered to be the same pedagogic case as those arising from dragging point D, as illustrated in Figure 6.13.

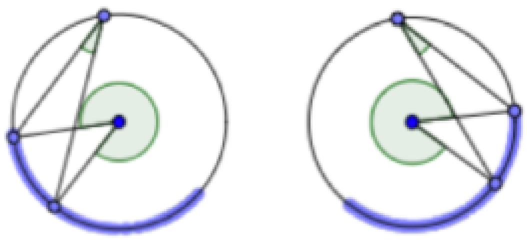


Figure 6.13 Points C and D are symmetrical so these two cross-quadrilateral configurations represent the same pedagogic case.

Ultimately this led to the compilation of a complete, systematic list of pedagogic cases as illustrated in Figures 6.9 and 6.10. Following the compilation of the complete numerical list of pedagogic cases, the videos were reviewed and a final circle theorem case list produced for each case study teacher, as illustrated in Figure 6.8. A second circle theorem case list was also created for Robert (see 'Robert*' in Figure 6.8) presenting the pedagogic cases as they arose through Robert's dragging following the second question 'how would you drag it for pupils?' Robert was unique in that his response to the second interview question 'how would you drag it for pupils?' was intentionally different from his response to the first question 'what's going on in this diagram?' Anne, Edward and Michael responded to the second interview question indicating they would drag D1 in a similar way to their response to question 1, before continuing their dragging to provide cases *in addition* to those already shown. This suggests that for these three teachers, listing cases in the order they arose in the interview also provides a reasonable indicator of how they would demonstrate D1 to pupils. Initially, Robert behaved more like a 'developer' of pedagogic resources, dragging the diagram mainly for his own exploration or to demonstrate something to the author as if to a colleague. His response to the second question was to start afresh, dragging D1 to his chosen starting configuration and dragging to show cases *instead of* those already shown. This suggests his response following the second question might be a better indicator of how he would demonstrate D1 to pupils. Whilst the case list 'Robert' provides a record of the interview, the case list 'Robert*' is a reasonable indicator of how he would demonstrate D1 to pupils and thus provides a fairer comparison with the case lists of the other three case study teachers.

6.3 Conclusion

The conceptualisation presented in the previous section demonstrates the complexity of mathematical knowledge for teaching circle theorems using technology. The complexity of this conceptualisation is paralleled by the complexity of knowledge in relation to arithmetic revealed by Ma's (1999) study, which led her to define a *profound understanding of fundamental mathematics*. The tabular list depicted in Figure 6.10 comprises 23 pedagogic cases, rising to 46 if 180-degree rotations are included. Memorising all these cases is unnecessary since it is possible to reconstruct the tabular list in Figure 6.10 given explicit knowledge of the eight main configurations; knowledge that the reflex angle at the centre represents a different pedagogic case to displaying the angle less than 180 degrees at the centre; and knowledge that dragging point B creates a different set of examples to dragging either point C or D. Nevertheless, even this amount of knowledge seems unrealistic for all teachers to acquire, especially if a similar level of complex knowledge is needed for all aspects of the curriculum. Adopting a distributed view of knowledge might offer potential strategies for supporting teachers to make coping with the complexity of mathematical knowledge for teaching using technology in general a more realistic proposition.

The topic of circle theorems provided an area of mathematical content that the case study teachers would be likely to be familiar with in the context of technology (Ruthven et al, 2008). This meant that a general lack of familiarity with the technological context, which would prevent inferences about teacher knowledge from being drawn, was unlikely. As discussed below, Anne exemplifies where a lack of general familiarity with the software arguably prevented access to her mathematical knowledge for teaching using technology. The topic of circle theorems can also be argued to represent the apex of geometry in the compulsory English school mathematics curriculum, since it is the topic where geometric proof is traditionally introduced. Thus it was likely to be a topic that challenged the case study teachers' mathematical knowledge. This was important to create an opportunity to observe and reflect on mathematical knowledge for teaching using technology, since as Putnam et al (1992) assert, the necessity of knowledge becomes apparent in its absence. Finally, the Knowledge Quartet provides a post-hoc justification for the highly complex conceptualisation of knowledge, presented in section 6.2, that forms a basis for informing and justifying a teacher's *choice and use of examples* and *decisions about sequencing*. This post-hoc justification alongside Mason and Watson's (2005; 2006) research on example spaces suggests that the approach taken here, i.e. mapping out a complete set of pedagogic cases and identifying a preferred pedagogical sequencing, may provide a means for conceptualising mathematical knowledge for teaching using technology for other areas of the mathematics curriculum. In particular, the conceptualisation presented in section 6.2 provides a basis for suggesting ways in which such knowledge could be measured. It is this type of conceptualisation, that underpins the construction of test-items, that is currently under-developed in existing measures of mathematical knowledge for teaching (e.g. Baumert et al., 2010; Hill et al., 2005; Tatto et al., 2012) as argued in Chapter 2. The next paragraph suggests ways in which mathematical knowledge for teaching using technology in relation to the topic of circle theorems could be measured. The chapter concludes by speculating tentatively about the relationship (if any) between a connectionist versus transmissionist teaching orientation and mathematical knowledge for teaching circle theorems using technology. These post-hoc inferences should be treated with caution and will require further research.

The development of the circle theorem case list suggests teachers' mathematical knowledge for teaching circle theorems using technology may be summarised as a two-dimensional measure, with one dimension providing an indicator of teachers' *choice and use of examples* and the other dimension providing an indicator of their *decisions about sequencing*. The total number of pedagogic cases the case study teacher elicited in the interview (column T in Table 6.2) provides a numerical summary as one possible indicator of teachers' *choice and use of examples*. Another possible indicator of teachers' *choice and use of examples* is the percentage of explicitly recognised pedagogic cases that teachers suggested they would show their pupils (column % in Table 6.2). Initiating a demonstration of D1 for pupils by (i) dragging point B or alternatively (ii) points C or D to elicit a new configuration provides a potential dichotomous indicator of teachers' *decisions about sequencing*. These potential indicators suggest the conceptualisation of mathematical knowledge for teaching circle theorems using technology could be operationalised in the form of an item asking teachers to provide all the configurations of the angle of the centre theorem that they can think of, possibly with the technological support of a diagram such as D1, with a follow up question asking which of these configurations they would select to show pupils and how they would sequence this selection of configurations if presenting them to pupils. The relevance of these potential indicators is discussed below along with a speculation as to their relationship or otherwise with teacher orientation in terms of the transmissionist measure of self-reported pedagogic practice.

Table 6.2 Possible quantitative indicators of the case study teachers' choice and use of examples

	transmissionist measure	**T**	**E**	**P**	**%** $= \frac{P}{E} \times 100$
Robert*	-1.01	12	8	6	75
Anne	-0.50	10	8	3	38
Edward	+0.74	10	8	4	50
Michael	+1.01	15	13	5	38

T = total number of pedagogic cases elicited in interview by case study teacher (implicit *and* explicit *excluding* direct prompts)

E = number of explicitly recognised pedagogic cases (explicit only *including* direct prompts)

P = number of pedagogic cases the teacher would positively show pupils

% = percentage of explicitly recognised pedagogic cases the case study teacher would positively show pupils (rounded to nearest whole number)

Note:. Robert* is presented here as a fairer comparison with the other case study teachers (see previous section) though the figures for Robert and Robert* were actually the same.

The total number of pedagogic cases the case study teacher elicited in the interview is a plausible indicator because it may be the mathematical knowledge made available in the classroom, i.e. including not only explicitly planned examples but also those arising through contingency and those left implicit, that has most impact on the quality of pupils' mathematical experience in the classroom. That is, it may be the knowledge distributed across both people and technology that has the most impact on the quality of pupils' mathematical experience in the classroom rather than the knowledge that resides in the individual teacher. The total number of pedagogic cases the case study teacher elicited in the interview shows no relationship with transmissionist measure, albeit acknowledging the problems of making any reliable inference given the small size and selective nature of the sample. The most transmissionist teacher, Michael, elicited the highest total number of pedagogic cases (15) of the four case study teachers, with the most connectionist teacher, Robert*, eliciting a total of 12 pedagogic cases - see Table 6.2. Occupying more intermediate positions on the scale of transmissionist self-reported pedagogic practice, Anne and Edward both elicited 10 pedagogic cases in total. The lack of relationship between the total number of pedagogic cases elicited in interview and teacher orientation is perhaps unsurprising given the findings of Askew et al (1997) that all but one of their measures of teacher knowledge were unrelated to teacher orientation. In particular, the only variable Askew et al (1997) found to be related to teacher orientation was *depth*, the percentage of links that were explained by the teacher in conceptual terms. The type of articulated conceptual knowledge suggested by the *depth* variable cannot be inferred from the total number of pedagogic cases elicited, since this number includes both pedagogic cases that were recognised explicitly and implicitly in the interview. This might offer some explanation as to why the total number of pedagogic cases elicited in interview appears unrelated to teacher orientation.

The percentage of explicitly recognised pedagogic cases that teachers suggested they would show their pupils (column % in Table 6.2) provides another possible indicator of teachers' *choice and use of examples*. When the case study teachers made a deliberate choice to include a particular pedagogic case in a demonstration to pupils, it indicated they could articulate a conceptual linkage to justify how the case would contribute to their pupils' understanding of the angle at the centre theorem. In addition to this conceptual linkage, a form of cost-benefit analysis appears to be included in this measure, perhaps similar to Ruthven's (2009) *time economy*. Transmissionist teachers might be more likely to place greater emphasis on the cost; connectionist teachers more on the benefit of showing pupils a particular pedagogic case, hence this measure might show some relationship to transmissionist measure. None of the teachers made a deliberate choice to exclude a pedagogic case. This is important because it suggests that the case study teachers believed there might be some benefit in a pedagogic case (even if they weren't quite sure what it was) and even if they didn't think it was sufficient to want to deliberately demonstrate that case to their pupils. Lack of inclusion tended to indicate that the teachers either did not articulate a conceptual linkage to justify how the case would contribute to understanding the theorem; that the teachers suggested they might delay (indefinitely) introducing the case due to instrumental concerns that it might make the theorem harder to 'spot' or that it might confuse low-attaining pupils. Thus the percentage of explicitly recognised pedagogic cases that teachers choose to show their pupils might bear some similarity to the *depth* variable in that it is an indicator of the percentage of conceptual links the case study teachers articulated, and thought worthwhile from a pedagogic viewpoint, between configurations of the angle at the centre theorem. Hence this measure of the case study teachers' choice and use of examples may be more likely to show a moderate relationship with teacher orientation. The figures in Table 6.2 provide tentative support for this speculative assertion, again given the small size and selective nature of the sample. The most connectionist teacher, Robert*, shows by far the highest percentage (75%) of explicitly recognised cases he suggested he would show pupils. A more transmission-oriented teacher, Edward, has a lower percentage of cases he would show pupils (50%) and Michael, the most transmissionist teacher, has the joint lowest percentage (38%). The exception to this pattern is Anne, a more connectionist-oriented teacher, who shares the joint lowest percentage (38%) with Michael. Anne was the least confident in using digital technologies. Unfortunately, she was also the only case study teacher not to receive the GeoGebra file in advance of the interview. These mitigating factors may offer an explanation of why Anne appears to have a low percentage of cases she would show pupils despite her connectionist orientation.

Initiating a demonstration of D1 for pupils by (i) dragging point B or alternatively (ii) points C or D to elicit a new configuration provides a potential dichotomous indicator of teachers' *decisions about sequencing*. This is important because dragging point C or D provides numerical variation to appreciate the doubling nature of the relationship between the angles in the theorem, whereas dragging point B does not. In addition, whether dragging was initiated with point B or points C/D tended to indicate how the convex quadrilateral configuration first occurred. For example, Michael's circle theorem case list shows that the convex quadrilateral configuration first occurred as pedagogic case 12, i.e. as a consequence of dragging point C or D, thus appearing as a natural continuation of numerically varying the angle at the centre to include reflex angles. In contrast, the circle theorem case lists of the other three case study teachers (Robert*, Anne and Edward) show the convex quadrilateral configuration first occurred as pedagogic case 5, i.e. as a consequence of dragging point B. This is a less preferable sequence of configurations since the convex quadrilateral configuration appears as a deviant example or "complication", reinforcing an impression of the arrowhead configuration as the standard or iconic configuration. This indicator of teachers' *decisions about sequencing* also shows no relationship with transmissionist measure.

Michael was the most transmissionist teacher. He was also one of the least experienced case study teachers, being only in his second year of teaching. In particular, he appeared to be the least mathematically confident of the four teachers and, with his undergraduate and masters degrees in economics, was hoping to transition to becoming an economics teacher. It is therefore surprising, despite the lack of relationship with transmissionist measure, that Michael's circle theorem case list should be the only one to indicate the preferred sequence of dragging to elicit the convex quadrilateral configuration first through dragging point C or D. In addition, Michael also has the highest total number of pedagogic cases the case study teacher elicited in the interview. Based on these indicators, Michael's circle theorem case list would be pedagogically preferable over those of the other three case study teachers since it maximises the mathematical knowledge made available (in the interview) as well as providing a better sequencing of configurations. One means of explaining this unexpected result is by viewing knowledge as distributed over both people and technology (Hutchins, 1995). Thus, in the context of the interview, Michael was able to demonstrate D1 to create a pedagogically preferable circle theorem case list because the knowledge made available through his demonstration was distributed more effectively both across the GeoGebra software and the author than the other case study teachers. On the other hand, although the knowledge made available in Michael's interview was more effectively distributed in terms of the number of configurations he elicited, arguably Robert, in particular, was better able to take pedagogic advantage of the knowledge distributed in his interview. Robert elicited the second highest total number of pedagogic cases, slightly fewer than Michael, however he was able to see and justify pedagogic value in a far higher percentage of the cases he recognised explicitly.

A distributed view of knowledge explains how teaching resources such as D1 could augment the knowledge made available in the classroom by supporting even a relatively inexperienced and mathematically unconfident teacher, such as Michael, to elicit a greater range of examples and sequence them to greater pedagogic effect. A teacher with relatively strong mathematical knowledge for teaching, such as Robert, would be better able to take pedagogic advantage of the greater range of examples made available through using teaching resources such as D1. This provides possible support for the interaction effect between PCK and the quality of tasks observed by Baumert et al (2010). In addition, as noted in the opening paragraph of this section, adopting a distributed view of knowledge might offer potential strategies for supporting teachers for coping with the complexity of mathematical knowledge for teaching using technology. Chapter 7 explores to what extent the mathematical knowledge made available through a teachers' interaction with technology is distributed across the individual teacher and the technology.

6.4 Summary

This chapter used the Knowledge Quartet as an analytic tool to provide a fine-grained analysis of the content of mathematical knowledge for teaching using technology in relation to the topic of circle theorems. In particular, this analysis revealed the highly complex nature of mathematical knowledge for teaching using technology, through the conceptualisation of such knowledge in relation to the topic of circle theorems. Addressing RQ2a, the centrality of the Knowledge Quartet codes *choice and use of examples* and *decisions about sequencing* provided a post-hoc justification for the development of a *circle theorem case list* for each case study teacher. The circle theorem case list allowed a comparison of both the specific circle theorem cases arising and the sequence in which they arose during the interview, providing a basis for suggesting ways in which mathematical knowledge for teaching using technology in relation to the topic of circle theorems could be measured. The chapter concluded by speculating tentatively about the relationship (if any) between transmissionist measure and mathematical knowledge for teaching circle theorems using technology. These post-hoc inferences should be treated with caution and will require further research.

Chapter 7 - Mathematical knowledge for teaching using technology as distributed: how and to what extent

Chapters 5 and 6 concentrated on the nature and content of individual teachers' own knowledge in relation to teaching mathematics using technology. In Chapter 5, Noss and Hoyles' (1996; & Kent, 2004; & Pozzi, 2002) notion of *situated abstraction* seemed appropriate as a description of mathematical knowledge for teaching using technology. This notion, along with Adler's (1999; 2001) concepts of the dilemma of transparency and visibility/invisibility, was useful in negotiating the apparent duality in seeing this knowledge simultaneously as abstract, mathematical knowledge and yet as *situated* in the context of teaching using technology. The data analysis reported in Chapter 6 produced a conceptualisation of mathematical knowledge for teaching using technology in relation to the topic of circle theorems. This conceptualisation indicates the highly complex nature of mathematical knowledge for teaching using technology. Chapter 6 concluded by suggesting that adopting a distributed view of knowledge might offer potential strategies for supporting teachers to cope with the highly complex nature of mathematical knowledge for teaching using technology.

Hence, in this chapter, the research focus shifts from individual teachers' own knowledge to how this knowledge is involved in the participatory relationship with technology (Remillard, 2005). Chapter 1 justified the use of Hutchins' (1995) view of distributed cognition as a framework for understanding how individual teachers' knowledge is involved in interacting with technology to produce the mathematical knowledge made available in the classroom. Detailing how and to what extent knowledge is distributed across teacher and technology is a means of describing the participatory relationship (Remillard, 2005). Thus this chapter addresses Research Question 2b, identified through the literature review in Chapter 2:

RQ2b *To what extent is the mathematical knowledge made available through a teachers' interaction with technology distributed across the individual teacher and the technology?*

The review of literature in Chapter 2 reported that most research on teacher knowledge in general and in mathematics education specifically, whilst recognising the situated nature of knowledge, remains underpinned by essentially individualistic assumptions

(Putnam & Borko, 1997). In particular, research on teacher knowledge informed by views of cognition as *distributed* (Hutchins, 1995) across persons and tools remains underdeveloped (Putnam & Borko, 1997). In research on technology in mathematics education, the instrumental approach appears commensurate with Hutchins' (1995) distributed view of cognition in the sense that, by considering an instrument as the combination of an artefact with the user's schemes of use, knowledge appears distributed across both teacher and technology. However, as noted in Chapter 1, the instrumental approach has a tendency to focus on teachers' learning, which it is not the primary focus of this study. Instead, this study concentrates on teachers as workers where the focus is on the knowledge they 'have', using technology in their work of teaching mathematics. Hence Hutchins' (1995) view of distributed cognition seemed more appropriate to this study because it focuses on analysing the interaction of humans with artefacts in the workplace.

Where a distributed view of cognition has been advocated, it remains unclear to what extent knowledge for teaching (using these resources) would necessarily be internal to the teacher and to what extent it could be said to reside in the resource itself. For example, Putnam and Borko (1997, p. 1287) suggest teachers might augment their pedagogical thinking through making "judicious use" of new information technologies available via the Internet or existing technologies such as textbooks (see also Putnam & Borko, 2000). Making judicious use of such technologies suggests teachers might require a thorough, *apriori* knowledge of these technologies for the purposes of teaching. Thus it remains unclear to what extent such the technology could augment teachers' thinking i.e. what part of knowledge for teaching could be considered as being distributed across the resource.

In contrast, Hutchins (1995) is careful to avoid suggesting tools augment human cognition, suggesting instead that tools transform a task by taking on some cognitive attributes. Critically, Hutchins states that his view of distributed cognition assumes that cognition does not only occur within an individual person, but also occurs through human interaction with artefacts and other humans and, in particular, that cognition partially resides in tools (see Chapter 1). In addition, his specification of minimum knowledge requirements necessary for an individual to carry out a computational task in interaction with an artefact provides an indication of *how* and to what extent cognition may be distributed across both human and artefact.

This chapter applies Hutchins' (1995) mode of argument to the case study teachers' suggestions of how they would use the GeoGebra diagram D1 to demonstrate the angle at the centre theorem to pupils. A framework for identifying instances of distributed cognition, where the mathematical knowledge made available was more or less distributed across the technology, was developed from an analysis of the circle theorem case lists presented in Chapter 6. In particular, the development of this framework led to the specification of minimum knowledge requirements necessary for the case study teachers to produce the circle theorem case lists in interaction with the GeoGebra software, the author and teaching-colleagues. By specifying minimum knowledge requirements, an indication is provided of how and to what extent the case study teacher's mathematical knowledge for teaching was distributed across the software, the author and teaching-colleagues. The minimum knowledge requirements provide a means of explaining why case study teacher Michael was able to produce a better circle theorem case list in pedagogical terms than the other more experienced and more mathematically confident case study teachers. This provides convincing evidence towards a distributed view of cognition. In particular, it suggests that an individual teacher 'having' more knowledge does not necessarily equate to an improvement in the mathematical knowledge made available at least in the interview setting. It seems reasonable to speculate that this might also generalise to a classroom setting. Finally, data from classroom observations of the case study teachers' use of technology are used to provide brief indications of how the framework might be exemplified in mathematics teachers' classroom practice.

7.1 Mapping the elements of Hutchins' description of distributed cognition to mathematics education

Hutchins (1995) argues that cognition is a socially situated cultural process that is distributed between humans, tools and the settings and environment within which tasks take place. In Hutchins' description of cognition as computation of navigational tasks distributed over humans in interaction with tools and each other, the humans were members of the navigation team on the USS Palau, an amphibious helicopter transport in the US Navy. The members of the navigation team had specific and well-defined roles, such as the pelorus operator, the bearing recorder and the plotter, which they carried out according to very strict regulations to be expected in a military setting. The computation of navigational tasks involved carrying out well-established routines that, although very strictly regulated, still left room for interpretation and improvisation. Hutchins lists amongst the various tools the navigation team use to carry out their tasks the alidade, the bearing log, the hoey (a one-arm protractor), the chart and so on. He argues these tools have cognitive attributes as they literally contain a repository of navigational knowledge, embedded in their construction. In particular, Hutchins argues these tools provide two things simultaneously (p.154). Firstly, they change the nature of the task. In Hutchins' terms, the computation of the navigational task is achieved through the "propagation of representational state" rather than through direct calculation. For, example, the nautical slide rule and nomograph permit the navigator to avoid algebraic reasoning and arithmetic in distance-rate-time problems by aligning indices with numbers on scales, or imagining numerical representations and making simple transformations of them. Secondly, these tools provide constraints on the organisation of action, in the sense that they regulate behaviour in such a way that the propagation of representational state that implements the computation can take place (Hutchins, p.154). For example, the nautical slide rule and nomogram preclude mistakes resulting from incorrect algebraic transformations taking place in the calculation of distance-rate-times since the relations $D = RT$, $R = D/T$, and $T = D/R$ are built into their physical structure. The environment in which the ship was embedded, in terms of the harbour, the landmarks, the sea-bed and the stars, also plays a critical role in organising and transforming the computational tasks of the navigation team.

Mapping the elements of Hutchins' description of distributed cognition, i.e. humans, tools etc, provides a means of contextualising his description in relation to this research project in particular and mathematics education research more generally. In terms of the semi-structured GeoGebra interviews, the setting was a discussion for research purposes between the author and the case study teacher based around a task involving knowledge of mathematics pedagogy in a technological context i.e. how they might use diagram D1 to demonstrate the angle at the centre theorem to their students. As in Hutchins' description, the setting is important because it describes the context in which the task was situated. The case study teachers were currently employed teaching mathematics to students in English secondary schools. Circle theorems is a topic often taught in year 10 or 11 since it is included in GCSE mathematics examination syllabi[4] and represents perhaps the most advanced topic in geometry in the compulsory English National

[4] At higher tier only, see for example the Edexcel (2012) syllabus.

Curriculum (QCA, 2007) for mathematics. For example, the topic of circle theorems may provide students' first (and only) encounter with geometric proof. Hence the case study teachers' considerations about how D1 might be used to demonstrate the angle at the centre theorem were to some extent shaped by knowledge of their own students and their perception of their students' needs in relation to the GCSE examination. The environment of the GeoGebra interviews tended to be in a mathematics classroom at the case study teacher's school that was not being used for teaching at that time. The author's laptop with mouse attached was arranged on a desk so that both the author and the case study teacher could comfortably see the screen and use the mouse to manipulate D1, enabling collaboration on the task.

The humans involved in the GeoGebra interviews were the author, the case study teacher, the case study teacher's students and the case study teacher's departmental colleagues. In the GeoGebra interviews, the students and the case study teacher's departmental colleagues were only hypothetically involved. There are two senses in which the students were hypothetical. Firstly and most obviously, no students were actually present during the interview. Secondly, the case study teachers were not planning a demonstration that was intended for use with actual students in an actual lesson. Due to the setting in terms of a discussion for research purposes, the case study teachers were discussing a demonstration of D1 for hypothetical students. It does however seem reasonable to assume that these hypothetical students would be largely based on an amalgam of those that the case study teachers actually taught. Finally, except in the case of Anne, the GeoGebra file was emailed in advance of the semi-structured interview and the case study teachers were encouraged to explore it beforehand and share it with their departmental colleagues. Michael at least showed the GeoGebra file to his Head of Department, mentioning this at the start of his interview. Again, the case study teacher's departmental colleagues were not present during the interview. Nonetheless, at least in Michael's case, they had a role to play in the performance of the interview task.

In comparison to members of the navigation team in the military setting of the USS Palau, the roles of the human participants in the interview setting were relatively fluid, in that they weren't laid down according to strict regulations. Nevertheless, there were expectations in relation to these roles. As teachers of mathematics in English secondary schools, expressing some confidence in their use of ICT, the case study teachers were expected to be able to participate in the interview task, to the extent they had planned and taught lessons on circle theorems before and would therefore be likely to have at least some familiarity with a dynamic diagram like D1 and possibly have even used something similar in their own lessons. For example, both Robert and Michael commented that they had used similar diagrams in their previous teaching. Edward had made brief notes on the GeoGebra file before the interview and his comments during the interview echoed these notes. Anne commented in the interview that she had never used such a diagram before, “I’ve never done this myself like this”; however she also noted that “There’s a lot of this around, you know, done by other ... we always use them, I share them” [Anne-GGb-intA, 28.6.2012]. In particular, since they had an opportunity to explore the GeoGebra file before the interview, albeit with the exception of Anne, it seemed reasonable to assume that planning a demonstration using D1 represented a routine task for the case study teachers, which they would be able to perform despite the unusual interview setting. Similarly, as teachers of mathematics in English secondary schools, the case study teacher’s departmental colleagues were expected to be able to discuss and offer advice on using D1 as a teaching resource for mathematics. Although these colleagues were not actually present in the interview, the cognitive processes involved in performing the interview task could potentially be distributed across them as human participants, due to the possibility of their involvement in discussing the GeoGebra file with the case study teacher beforehand.

The author’s role was as an interviewer interested in mathematics teachers’ use of technology for research purposes. However, a critical part of this role in shaping both the nature and content of the interview was the author’s own experience of teaching mathematics in English secondary schools. Thus at times, the case study teachers appeared to perceive the author to some extent as a colleague, though clearly one without the contextual knowledge of their departmental colleagues in relation to teaching mathematics at their particular school. For example, Robert appeared to speak on equal terms with the author in discussing the design of D1. Michael on the other hand tended to treat the author as a senior colleague, commenting off-record on the similarity between conversations with his mentor in initial teaching training. This was an indicator both of his lack of experience and mathematical confidence in relation to teaching compared with the other case study teachers. Despite the difference in roles, it therefore seems reasonable to assume that the cognitive processes involved in performing the interview task might be distributed across the author and the case study teacher’s departmental colleagues in similar ways.

The case study teacher's students were hypothetically expected to cooperate with the planned demonstration using D1, in accordance with the norms of a mathematics classroom. Again, although the students were not present in the interview, the cognitive processes involved in performing the interview task could potentially be distributed across them as human participants, since the case study teachers were likely to base their planning on expectations of their mathematical behaviour in lessons gleaned from past classroom experience. However, how and to what extent cognitive processes were distributed across students seemed likely to be qualitatively different from how and to what extent they were distributed across either the author or the case study teacher's departmental colleagues, by virtue of the significant difference in the roles these human participants play in the performance of the interview task.

The tool that is the main focus of the analysis presented in this chapter is the diagram D1, constructed in GeoGebra. Like the tools in Hutchins' description of navigation, D1 provides two things simultaneously. Firstly, it changes the nature of the task. That is, using D1 changes the cognitive processes necessary for a teacher's demonstration of the angle at the centre theorem to display a pedagogically desirable *choice and use of examples* and *decisions about sequencing*. For example, D1 permits the teacher to generate examples of the angle at the centre theorem by dragging any one of points B, C or D. This avoids having to generate and construct examples from 'scratch'. In addition, these examples are not generated at random. Instead, they are produced to give the appearance of continuous dynamic variation according to the construction of the diagram following the rules of Euclidean geometry embedded in the software through its design. This has the effect of sequencing the examples produced in particular ways, since only one point may be dragged at a time, to provide a pedagogic structuring of examples whether intended by the teacher or not. Indeed on opening the GeoGebra file, the starting configuration of D1 introduces limits on the *decisions about sequencing* that are possible – see for example Figure 6.9 in Chapter 6.

Secondly, D1 provides constraints on the organisation of action. The construction of D1, according to the geometric rules programmed into the software, limits the mathematical variation possible through manipulation of the diagram. In general, limiting the dimensions of mathematical variation seems to be pedagogically desirable, to allow instances of mathematical invariance such as the angle at the centre theorem to be apprehended. If everything is changing no invariance may be distinguished. Thus some of the case study teachers were less in favour of using diagrams D2 and D3 to introduce the angle at the centre theorem because there were too many dimensions of mathematical variation, whether they articulated it in quite these terms or not. In this sense, the constraints provided by D1 regulate behaviour in such a way that a demonstration of the angle at the centre theorem can take place. For example, dragging D1 will not produce non-examples of the angle at the centre theorem (ignoring rounding errors). Similarly, as noted above, dragging D1 will not produce a random sequence of examples. Instead, examples are generated in a particular way, coordinating aspects of geometric and numerical variation so that mathematical invariance may be apprehended.

In terms of the classroom observations of the case study teachers' use of technology, the setting was a mathematics lesson on a topic and with a group of students of the teacher's choice. The task was to teach an actual lesson using some form of digital technology, rather than planning a hypothetical one. The human participants were again the author, the case study teacher, the case study teacher's students and the case study teacher's departmental colleagues. Clearly, the roles played by the participants were somewhat different in this setting. Most obviously, the students were present and not at all hypothetical as in the GeoGebra interviews. The author was observing the lesson, suggesting a more passive role rather than the active participation required in the GeoGebra interviews. The roles of the case study teacher and their departmental colleagues were still as teachers of mathematics in English secondary schools. Again, although the case study teachers' colleagues were not actually present in the lesson, the cognitive processes involved in performing the interview task could potentially be distributed across them as human participants, due to the possibility of their direct or indirect involvement in planning the lesson beforehand. The case study teachers employed a range of digital and non-digital tools in their IWB and computer suite lessons, whose use served to transform and organise the task of teaching the mathematical topic to their students. The environment varied according to where the lesson took place. For Robert and Michael's IWB lessons, the environment was their normal mathematics classroom. The other observations required room changes so that the teachers were conducting their lessons in a relatively unfamiliar *working environment* (Ruthven, 2009), requiring adaptations of their normal classroom routines.

7.2 A framework for identifying instances of distributed cognition

This section describes a two-dimensional framework for identifying instances of distributed cognition, developed from an analysis of the circle theorem case lists presented in Chapter 6. In particular, this framework indicates to what extent cognition could be considered as residing in the individual case study teacher or as distributed over tools and other human participants in the interview. The justification for these assertions is provided in the following section through the specification of minimum knowledge requirements necessary for the case study teachers to produce the circle theorem case lists in interaction with the GeoGebra software, the author and teaching-colleagues.

7.2.1 Outlining the framework

The circle theorem case lists are a representation of the mathematical knowledge made available in the semi-structured GeoGebra interviews through the case study teacher's manipulation of D1. More specifically, they represent the *choice and use of examples* and *decisions about sequencing* made by the case study teacher in interaction with D1 and the other human participants in the interview. The *choice and use of examples* and *decisions about sequencing* made in the interview is the observable outcome of cognitive processes. These processes are assumed to be distributed over the case study teacher, the tool D1 and the other human participants, involved in the performance of the task of suggesting how diagram D1 might be used to demonstrate the angle at the centre theorem to the case study teacher's students. As described in the previous section, the other human participants include the author, the case study teacher's departmental colleagues and the case study teacher's students. Thus instances of distributed cognition might be identified through analysing the *choice and use of examples* and *decisions about sequencing* the case study teacher made in interaction with (a) the diagram D1, (b) their students, and (c) the author or their departmental colleagues, as represented by their circle theorem case list.

The three categories (a) technology, (b) their students, and (c) the author or their departmental colleagues, therefore form the first dimension of the framework for identifying instances of distributed cognition. For example, from the circle theorem case lists, pedagogic cases that were coded as being directly prompted by the author were regarded as instances where cognition was distributed to some extent over the author in interaction with the case study teacher and the technology (see Chapter 6, Table 6.1 for coding system). Pedagogic cases that were coded as indirectly prompted or unprompted were regarded as instances where cognition was distributed to some extent over the technology i.e. diagram D1. The categorisation along this dimension might appear to introduce an artificial separation between instances of cognition distributed across technology and the other human participants in interaction with the case study teacher. These examples suggest that the categories should not be seen as mutually exclusive i.e. the categories should not be assumed to indicate instances where cognitive processes were distributed solely over diagram D1 or solely over the author in interaction with the case study teacher, for example. Instead the categories should be taken to indicate instances where cognition was distributed mainly over diagram D1 (in interaction with the author) or mainly over the author (in interaction with D1) and in interaction with the case study teacher. Since the focus of this research project was primarily on technology, instances of distributed cognition arising from the data in the categories involving the other human participants in interaction with the case study teacher are relatively limited. In particular, instances of distributed cognition arising from the data involving the case study teacher's students or their departmental colleagues are especially limited, since no participants from either category were present in the GeoGebra interviews. Nevertheless, occasionally, where a case study teacher mentioned their students or their departmental colleagues, this provided the basis for a thought experiment hypothesising the kind of instance of distributed cognition that might have occurred had these participants been present.

The second dimension of the framework signifies to what extent cognition could be inferred as distributed across the GeoGebra diagram D1 as opposed to residing in the individual case study teacher. In other words, the second dimension of the framework indicates to what extent cognition could be inferred as more or less distributed across technology in interaction with the case study teacher and the other human participants. The second dimension consists of two categories indicating instances of distributed cognition where the case study teachers' choice and use of examples and decisions about sequencing were either *articulated* or *unarticulated*, each of which is sub-divided into two further categories, *anticipated* and *unanticipated opportunities*. The system for coding instances of distributed cognition according to these categories is described below.

From the circle theorem case lists, an *articulated* choice and use of examples indicates the case study teacher commented verbally on the pedagogic case, whilst D1 was held stationary or left static showing the case, following the code *explicit recognition*, see Table 6.1 in Chapter 6. Thus an *articulated* choice and use of examples suggests a *writerly* response (Bowe et al., 1992) to a particular configuration of D1: a recognition that the configuration represents a distinct case from a pedagogic point of view, hence a choice needs to be made as to whether and how to use it. Similarly, an *articulated* decision about sequencing indicates the case study teacher commented verbally on their preference for one dragging sequence over another. Again an articulated decision about sequencing suggests a writerly response to D1: recognition that dragging D1 imposes a particular pedagogic structure or sequence of examples that is then available to critique. This makes no commitment to the quality of the case study teacher's critique or their choice regarding a pedagogic case. Thus it is important to clarify that although linking the articulated category with a writerly response suggests a positive example of foundational knowledge in some respects, it is neutral in terms of the case study teacher's transformation of mathematical knowledge for pedagogic purposes and the foundational knowledge underpinning that transformation. That is, linking the articulated category with a writerly response does not imply that the mathematical knowledge made available in the interview through an articulated choice and use of examples or decisions about sequencing would *necessarily* be any better (or worse) than an unarticulated choice and use of examples or decisions about sequencing - see also the discussion in Chapter 6 regarding to the Knowledge Quartet code *adherence to textbook*. It is also important to note that the *articulated* category is not intended to signify that the case study teacher provided a full description and/or justification of their choice and use of examples and decisions about sequencing, but rather that some verbal indication was provided.

An *unarticulated* choice and use of examples indicates the case study teacher made no verbal comment regarding the pedagogic case, despite eliciting it through dragging in the course of the interview, following the *implicit recognition* code, see Table 6.1 in Chapter 6. Thus an *unarticulated* choice and use of examples suggests a *readerly* response (Bowe et al., 1992) to a particular configuration of D1. That is, the case study teacher did not recognise that the configuration could represent a distinct case from a pedagogic viewpoint, hence there was no deliberate choice to articulate. Similarly, an *unarticulated* decision about sequencing indicates the case study teacher dragged D1 to elicit a particular sequence of pedagogic cases without verbally commenting on their pedagogical preference for this particular sequence. Again an unarticulated decision about sequencing suggests a readerly response to D1: a lack of recognition that dragging D1 imposes a particular pedagogic structure or sequence of examples, which as a result is unavailable to critique. Again it is important to clarify here that although linking the unarticulated category with a readerly response suggests a lack of foundational knowledge in some respects, it is neutral in terms of the case study teacher's transformation of mathematical knowledge for pedagogic purposes. That is, linking the unarticulated category with a readerly response does not imply that the mathematical knowledge made available in the interview through an unarticulated choice and use of examples or decisions about sequencing would necessarily be any worse (or better) than an articulated choice and use of examples or decisions about sequencing. In particular, in the next sub-section examples that could be interpreted as a positive choice and use of examples or decision about sequencing are provided in both the articulated and the unarticulated category. Similarly, examples that could be interpreted as a negative choice and use of examples or decision about sequencing are provided in both categories.

Anticipated opportunities indicate a choice and use of examples or decision about sequencing which appeared to be part of the case study teacher's routine response when presented with the task of suggesting how they might demonstrate the angle at the centre theorem using D1 to their pupils. *Anticipated opportunities* were identified as the case study teachers' *unprompted* choice and use of examples and decisions about sequencing i.e. as elicited through their dragging in response to the first two interview questions, see Chapter 6, Table 6.1. *Unanticipated opportunities* were identified as the case study teachers' *indirectly prompted* or *directly prompted* choice and use of examples and decisions about sequencing i.e. in as elicited through their dragging in response to the further interview prompts, see Chapter 6, Table 6.1. Thus *unanticipated opportunities* indicate a choice and use of examples or decisions about sequencing which did not appear to be part of the case study teacher's routine response, but rather a response to a contingent situation brought about by the author's unanticipated prompts. This suggests that instances of distributed cognition categorised as *unanticipated opportunities* might correspond to situations coded under the Knowledge Quartet supra-category of Contingency. Similarly, *anticipated opportunities* correspond to situations that might be coded as indicating Foundation, Transformation or Connection knowledge, since they indicate the knowledge made available through the case study teachers' routine performance.

7.2.2 Exemplifying the framework

Table 7.1 displays the framework for identifying instances of distributed cognition and exemplifies the categories using data from the semi-structured GeoGebra interviews where possible. In reading this section, the reader will find it helpful to have Figure 6.9, the numerical list of pedagogic cases from Chapter 6, to hand – see separate sheet provided in the back cover of the thesis. As noted in the previous sub-section, since the focus of this research project was primarily on technology, the categories involving technology are more fully exemplified with instances of distributed cognition than those involving the other human participants in interaction with the case study teacher. Table 7.1 also displays the four, second dimension categories in order, according to what extent cognition could be inferred as more or less distributed across technology in interaction with the case study teacher and the other human participants, as follows, starting with the most distributed:

- *unarticulated, unanticipated opportunities*
- *unarticulated, anticipated opportunities*
- *articulated, unanticipated opportunities*
- *articulated, anticipated opportunities*

This ordering is partially justified below through a discussion of the examples in each category of the framework. The specification of minimum knowledge requirements in the next section completes the justification.

Table 7.1 Framework for identifying instances of distributed cognition

Less distributed across technology and other human participants

More internal to the teacher

		Technology	Students	Colleagues
Unarticulated	Unanticipated Opportunities (KQ: Cy)	CUE and DS: Indirectly prompted, unarticulated e.g. Robert's sequence of cases 21-17-16-19		CUE and DS: pedagogic cases introduced due to the author's dragging e.g. Robert's interview 8-12'-6'
	Anticipated Opportunities (KQ: Fo, Tr, Co)	CUE: Robert, Anne and Edward elicited pedagogic case 3 was unprompted and unarticulated DS: Michael's sequence of cases 1-6-12; Anne's sequence of cases 1-3-5-4		DS: Michael's Head of Department might have advised him to drag D1 to produce the sequence of cases 1-6-12
Articulated	Unanticipated Opportunities (KQ: Cy)	CUE: Indirectly prompted, articulated e.g. all case study teachers elicited pedagogic case 10; Robert and Edward elicited pedagogic case 12, 13 DS: Robert and Edward choosing the sequence 1-5 before 1-6-12-13		CUE: Directly prompted, articulated – the author elicited the upside-down arrowhead i.e. either pedagogic case 1' or 6'
	Anticipated Opportunities (KQ: Fo, Tr, Co)	CUE: Unprompted, articulated e.g. Michael elicited pedagogic cases 1, 6, 12; Robert and Edward elicited 1, 5, 6; and Anne elicited 1, 5, 4 DS: Robert's decision to elicit the sequence of cases 1-5 before 1-6; Edward's decision to sequence 1-1 before 1-6	CUE: Robert's implication that students' dragging tends to elicit a pedagogically undesirable choice and use of examples	

CUE = choice and use of examples; DS = decisions about sequencing. Note where these codes appear under an *unanticipated* category, it indicates that a contingent moment led to a CUE or DS being made, whether this appeared deliberate or inadvertent at the time.

Unarticulated, unanticipated opportunities

Unarticulated, unanticipated opportunities are instances of distributed cognition where the cognitive processes involved in producing the case study teachers' circle theorem cases lists appear to be most distributed over technology and the other human participants. This is because, due to the case study teacher's *readerly* response to D1, the generation of pedagogic cases and their sequencing seemed to have been mainly delegated to the technology and other human participants rather than the case study teacher. Instances in this category, where cognition appears mainly distributed over technology, occurred when the case study teachers were dragging D1 in an exploratory way, exploring D1 for themselves perhaps, following an indirect prompt. Such exploratory dragging sometimes elicited and sequenced pedagogic cases of which (and in ways) the case study teacher appeared unaware, since they did not comment on them. Nevertheless the pedagogic cases were made available in the interview and so contributed to the case study teacher's circle theorem case list. For example, Robert's exploratory dragging of D1 produced a sequence of four pedagogic cases, 21 17 16 19, none of which he commented on. Neither did he comment on the dragging sequence needed to elicit these four pedagogic cases. Thus he seemed unaware of the pedagogic cases he elicited and hence the responsibility for generating these particular cases (as opposed to some other cases) appeared to rest mainly with the technology. The desirability of having these cases made available in the interview is dependent on their usefulness for pedagogic purposes, not on Robert's readerly response per se. Robert's exploratory dragging might represent a positive choice and use of examples if it raised useful cases that he wouldn't have elicited otherwise, even though they remained unarticulated. His exploratory dragging might instead be interpreted as a negative choice and use of examples if the cases distracted from the angle at the centre theorem, again even though they remained unarticulated.

Unarticulated, unanticipated opportunities where cognition appears mainly distributed over the author (in interaction with the technology), occurred when the author was dragging D1 deliberately to elicit the upside-down arrowhead configuration and in doing so inadvertently introduced and sequenced pedagogic cases. The author did not comment on these pedagogic cases because to do so would have distracted from the intended introduction of the upside-down arrowhead configuration. The author had also asked the case study teachers to “bear with me”, indicating they were not expected to follow the dragging sequence. The case study teachers did not comment on these inadvertently introduced pedagogic cases, and thus appeared unaware that they had been elicited. Hence again, responsibility over the generation of pedagogic cases and their sequencing seemed to rest mainly with the author in interaction with the technology rather than the case study teacher. For example, during Robert’s interview, the author elicited a sequence of three pedagogic cases (8 12’ 6’) whilst dragging D1 to introduce the upside-down arrowhead configuration. Robert did not comment on these cases or on their sequencing. Similarly, in each of the other case study teachers’ interviews, the author elicited one pedagogic case whilst dragging D1 (Edward and Anne 10’; Michael 21). Again, the case study teachers did not comment on the inadvertent case.

Unarticulated, anticipated opportunities

Unarticulated, anticipated opportunities are those where the case study teachers’ apparently routine response in terms of dragging of D1 elicited pedagogic cases or a sequencing of pedagogic cases that remained unarticulated. Robert, Anne and Edward all elicited the cross quadrilateral configuration (pedagogic case 3) through their routine of dragging point B to initiate a demonstration of the angle at the centre theorem, however none of them commented verbally on this case. This provides an example of an unarticulated anticipated *choice and use of examples*, where cognition appears mainly distributed over technology. Since their dragging appeared to be a routine response, the case study teachers seemed to be somewhat more in control of their generation of this pedagogic case than for an unarticulated, unanticipated opportunity. Nevertheless, their lack of articulation suggests a readerly response, i.e. that they were unaware of the pedagogic case they had elicited and hence the responsibility for generating this particular case (as opposed to some other unspecified case) appeared to rest mainly with the technology.

Two further examples of unarticulated, anticipated opportunities, where cognition appears mainly distributed over technology, are provided by Michael and Anne’s *decisions about sequencing*. Michael’s routine response produced the sequence of pedagogic cases 1-6-12, through dragging points C and D to initiate a demonstration of the angle at the centre theorem. At no point did Michael drag point B to disrupt the arrowhead configuration until directly prompted to by the author. Even then he avoided articulating a decision about sequencing by commenting on his apparent preference for dragging points C or D versus point B to initiate a demonstration of the angle at the centre theorem. Anne's routine response was to begin by dragging point B, producing the sequence 1-3-5-4. Although she later dragged points C and D without direct prompting from the author, at no point during the interview did she comment on her apparent preference for dragging point B to initiate a demonstration of the angle at the centre theorem. Michael and Anne's lack of an articulated preference for dragging either point B or points C/D to initiate their demonstration suggested they were unaware of the existence of an alternative to their initial dragging routine. Hence, they appeared unaware of their decisions about sequencing. Since their dragging appeared to be a routine response, the case study teachers seemed to be somewhat more in control of their decisions about sequencing than for an unarticulated, unanticipated opportunity. However, their apparent lack of awareness regarding their decisions about sequencing suggests a readerly response, which seems to imply the responsibility for generating the particular sequencing of cases (as opposed to some other sequencing) appeared again to rest mainly with the technology. Again, labelling Michael and Anne's decisions about sequencing as a readerly response does not imply a writerly response would have produced a better result. Indeed, as described below in the section on articulated anticipated opportunities, Edward and Robert's writerly response produced a sequence of pedagogic cases similar to Anne's readerly response and arguably less desirable from a pedagogical perspective than Michael's readerly response (see also Chapters 5 and 6).

There were no clear examples of unarticulated, anticipated opportunities, where cognition appears mainly distributed over the other human participants, in interaction with technology and the case study teacher. However, Michael's remark that he showed the GeoGebra file to his Head of Department provides the basis for a thought experiment as an example of an unarticulated, anticipated opportunity where cognition appears mainly distributed over his departmental colleagues. It is possible that Michael's Head of Department suggested to him that dragging C and D was a good way to use D1 to introduce the angle at the centre theorem and demonstrated the dragging sequence to produce pedagogic cases 1-6-12 to illustrate his/her point. Michael might have accepted this suggestion unquestioningly without need for an explanation, since it came from a senior teaching colleague (a readerly response), or he might simply have temporarily forgotten the explanation provided. In either case, his replication of this dragging sequence as a routine response, without articulating his decisions about sequencing, would provide an example of an unarticulated, anticipated opportunity where cognition appears mainly distributed over his departmental colleagues. Again this thought experiment underlines that knowing a good routine can sometimes provide a pedagogically desirable outcome i.e. a good decision about sequencing, even if a justification for doing so is (temporarily) not known. In addition, knowing a good routine might allow the teacher to avoid having to make explicit decisions about sequencing, affording an opportunity to focus elsewhere, for example, on making connections through a good choice and use of examples instead. In this sense, Michael was able to make a straightforward connection between the arrowhead and convex quadrilateral configurations to demonstrate the angle at the centre theorem holds for both angles less than 180 degrees and reflex angles that eluded Edward and to some extent Robert as well.

Articulated, unanticipated opportunities

Articulated, unanticipated opportunities form part of the case study teachers' evoked concept image (Tall & Vinner, 1981). As noted in Chapter 6, the case study teacher's evoked concept image is the set of explicitly known pedagogic cases (amongst other things), evoked both in and by the technological and interview context, that they can draw on to make a deliberate *choice and use of examples* or *decision about sequencing*. A deliberate choice and use of examples or decision about sequencing implies the case study teachers' made a *writerly* response to the pedagogic case or sequence of pedagogic cases they elicited through dragging. Articulated, unanticipated opportunities are those where the case study teachers' encountered pedagogic cases or a sequencing of cases, which were not part of their routine response, causing them to pause and comment verbally. At times such opportunities seemed more or less unexpected. For example, in response to indirect prompting by the author, Robert and Edward dragged D1 to elicit pedagogic case 13, where the segment CD forms a diameter so that D1 shows the angle in a semi-circle is right. This configuration seemed familiar to both Robert and Edward and so less unexpected. Indeed it might be reasonable to assume this case would usually have formed part of their planned demonstration for a real lesson. Nevertheless, it did not form part of their routine response in the interview situation. More unexpected was pedagogic case 10, a confusing orientation of the cross-quadrilateral configuration. This case caused both Robert and Michael, for example, to spend some time checking whether the angle at the centre theorem still held. Similarly, pedagogic case 12, the convex quadrilateral configuration, caused Edward some consternation, which he did not entirely resolve to his satisfaction. In these examples, as noted earlier in the paragraph, the case study teachers' articulation of the pedagogic cases allowed them to make a deliberate *choice and use of examples* i.e. a writerly response. Thus they appear more in control and correspondingly cognition seems less distributed across the technology and the other human participants. Nevertheless, since these examples were not part of the case study teachers' routine response, the responsibility for generating them still appears to lie to some extent with the technology. Here a writerly response can be seen to produce both positive and negative outcomes in terms of a choice and use of examples. For example, in a negative outcome of a writerly response, Edward chose to relegate a key pedagogic case due to a lack of foundational knowledge regarding the statement of the angle at the centre theorem. On the other hand, in a positive example of a writerly response, both Robert and Edward saw an opportunity to use pedagogic case 13 to make a connection with the angle in a semi-circle theorem. These examples could also be considered as instances where cognition also appears distributed to some extent across the author. However, since the case study teachers' dragging was in response to an indirect prompt, they were coded as articulated, unanticipated opportunities, where cognition appears distributed (mainly) over technology.

An example of an articulated, unanticipated *decision about sequencing*, where cognition appears mainly distributed over technology, is Edward and Robert's choice to introduce pedagogic case 5 before 12. Both Edward and Robert chose to drag B first, producing the sequence 1-5, and only later dragged points C and D to produce 1-6-13-12. This was an unanticipated decision about sequencing because, for both case study teachers, pedagogic cases 12 and 13 arose only after indirect prompting from the author. Edward stated he would drag B to elicit pedagogic case 5, as a first "complication", before continuing to drag points C and D to elicit pedagogic case 13, the angle in a semi-circle, and pedagogic case 12, the convex quadrilateral configuration displaying the correct angle at the centre. Similarly, Robert suggested he would drag B first, hoping his students would prompt him to move B onto the 'wrong' segment, thus eliciting pedagogic case 5. He suggested he would then continue his demonstration by dragging points C and D, presumably to elicit pedagogic cases 6, 13 and 12 as he had done earlier in the interview – see circle theorem case lists, Chapter 6. This instance is discussed further under the next category articulated, anticipated opportunities.

Articulated, unanticipated opportunities where cognition appears mainly distributed over the author (in interaction with the technology), occurred when the author dragged

D1 deliberately to elicit the upside-down arrowhead configuration. The pedagogic case was unanticipated, since none of the case study teachers had elicited it through their own dragging. When asked what they thought of this configuration, each of the case study teachers articulated why it could be seen as different to previous cases from the point of view of mathematics pedagogy, although Michael required some further prompting to do so. This prompted them to make a deliberate *choice and use of examples*, with Robert and Anne suggesting this was a case they thought worthwhile showing to their students, whilst Edward and Michael were more sceptical of its worth. The case study teachers made a writerly response to the upside-down arrowhead configuration and thus appear more in control of their choice and use of examples. Correspondingly, cognition seems less distributed across the author. Nevertheless, since this pedagogic case was not part of the case study teachers' routine response, the responsibility for generating them still appears to lie to some extent with the author, in interaction with the technology.

Articulated anticipated opportunities

Articulated, anticipated opportunities are where the case study teachers' encountered pedagogic cases, which caused them to pause and comment verbally, during what appeared to be their routine response to demonstrating the angle at the centre theorem using a diagram such as D1. They are instances of distributed cognition where the cognitive processes involved in producing the case study teachers' circle theorem cases lists appear to be least distributed over technology and the other human participants. This is because, due to the case study teacher's routine and *writerly* response to D1, responsibility and control over the generation of pedagogic cases and their sequencing seemed to rest mainly with the case study teacher rather than the technology and other human participants. For example, in their apparently routine response to using D1 to demonstrate the angle at the centre theorem, each of the case study teachers elicited and articulated two pedagogic cases beyond the starting configuration. In the interview context, some of these instances actually appeared more like an articulated, unanticipated choice of examples. However, it is possible to imagine that if the case study teachers had planned their demonstrations in advance, as they might for a lesson, then these would have been clear-cut instances of articulated, anticipated choice and use of examples. Thus it is reasonable to infer the case study teachers dragged D1 in anticipated ways to elicit pedagogic cases in a deliberate *choice and use of examples*. In these instances, to the extent cognition appears distributed, it appears mainly distributed over technology compared to the other human participants.

Robert and Edward also both made articulated, anticipated *decisions about sequencing*. As shown by their respective circle theorem case lists, both Edward and Robert began by dragging point B to elicit pedagogic case 5, the convex quadrilateral configuration displaying the 'incorrect' angle at the centre. Returning D1 to the starting configuration, they then went on to drag point C or D to elicit pedagogic case 6, demonstrating numerical variation whilst retaining the arrowhead configuration. Both later articulated their decisions about sequencing later in the interview. Robert stated he thought it was important to drag B first, showing the invariance of the angle on the circumference, to make a link with students' knowledge of the angles in the same segment theorem. In dragging B first, he hoped his students would prompt him to move B onto the 'wrong' segment, thus eliciting pedagogic case 5, to make the point that the angle at the circumference would change. He stated that he would then go on to drag points C and D to demonstrate the angle at the centre theorem. Thus he articulated a decision to drag D1 to elicit the sequence of cases 1-5 before the sequence 1-6. Edward later clarified that he would drag B first, showing the invariance of the angle on the circumference, restricting his dragging to the 'correct' segment. He would then drag points C and D, without disrupting the arrowhead. Thus he articulated a decision to drag D1 to elicit the sequence of cases 1-1 before the sequence 1-6. Edward stated he would only then drag B to elicit pedagogic case 5, before continuing to drag points C and D to elicit pedagogic case 13, the angle in a semi-circle, and pedagogic case 12, the convex quadrilateral configuration displaying the correct angle at the centre.

Robert and Edward made a writerly response to D1 in deciding to drag point B before dragging points C and D in what appeared to be their routine sequencing of cases. In addition, this decision partly led to an articulated yet unanticipated decision about sequencing to introduce pedagogic case 5 before 12. That is, both Edward and Robert chose to drag B first, producing the sequence 1-5, and only later dragged points C and D to produce 1-6-13-12. This was an unanticipated decision about sequencing because, for both case study teachers, pedagogic cases 12 and 13 arose only after indirect prompting from the author (see articulated, unanticipated opportunities). This appears to be a negative outcome of a writerly response, since showing the invariance of the angle on the circumference, by dragging B first, distracts attention from the angle at the centre theorem. Thus when pedagogic case 5 is introduced, the doubling relationship initially appears to have broken down and requires some work to re-establish that it still holds. Hence the convex quadrilateral configuration can appear to be a 'special case' rather than a standard configuration of the angle at the centre theorem. If instead points C and D are dragged first to elicit the sequence 1-6-13-12, numerical variation demonstrates the doubling relationship and the convex quadrilateral configuration appears as a natural consequence of allowing the angle at the centre to vary beyond 180 degrees. Thus Michael's readerly response, dragging to produce the sequence 1-6-12 (see unarticulated, anticipated opportunities), seems arguably more desirable than Edward and Robert's writerly response with regard to decisions about sequencing. In these instances, to the extent cognition appears distributed, Edward and Robert's articulated, anticipated *decisions about sequencing* appear mainly distributed over technology compared to the other human participants.

Early on during the interview, Robert demonstrated his impression of students' dragging when given a diagram like D1 to explore. He dragged the points B, C and D quickly in a jerky fashion, as if to imply that students' dragging is unsystematic and hence they fail to encounter key pedagogic cases such as the three standard configurations. This could be interpreted as an articulated, anticipated choice and use of examples: a recognition that some explicit pedagogic structure needs to be imposed in some way, either via a teacher demonstration or a structured worksheet for example, to ensure students encounter key pedagogic cases. In this instance, to the extent cognition appears distributed, Robert's articulated, anticipated *choice and use of examples* would appear mainly distributed over students in interaction with the technology.

7.3 Minimum Knowledge Requirements

This section proposes minimum knowledge requirements for a *choice and use of examples* and *decisions about sequencing* in each of the four categories of the second dimension of the framework for identifying instances of distributed cognition. The minimum knowledge requirements provide a means of explaining why case study teacher Michael was able to produce a better circle theorem case list in pedagogical terms than the other more experienced and more mathematically confident case study teachers (see the next section and also Chapter 6). By specifying minimum knowledge requirements, an indication is provided of how and to what extent the case study teacher's mathematical knowledge for teaching was distributed across the software, the author and teaching-colleagues. In particular, this provides evidence to support the assertions made in the previous section of the ordering of the four categories in terms of the extent cognition was distributed over technology and the other human participants, from most to least distributed. Where possible, the minimum knowledge requirements are also grounded in examples drawn from GeoGebra interview data to provide a context in which they can be understood.

This section begins by introducing notation to describe the knowledge requirements for a *choice and use of examples* and *decisions about sequencing*. The diagram below represents a *choice and use of examples* where M and N are pedagogic cases and *s* is a dragging sequence to obtain N from M.

$$\mathbf{M} \xrightarrow{s} \mathbf{N}$$

Thus, for example, an articulated *choice and use of examples* implies the case study teacher's recognition of N as a pedagogic case distinct from some M. Recognition of M is also implied, although this may simply be the starting i.e. arrowhead configuration. If it is to be repeatable, an articulated *choice and use of examples* additionally implies recognition of the dragging sequence *s*, i.e. pedagogic case N cannot be used again unless a dragging sequence is known to elicit it from some pedagogic case M. A teacher's articulation of a *choice and use of examples* is observable through their recorded speech. Their recognition of a pedagogic case or dragging sequence is unobservable: it is inferred from their articulation.

Similarly, the diagram below represents a *decision about sequencing* where M, N_1 and N_2 are pedagogic cases and s_1 and s_2 are dragging sequences. s_1 and s_2 are not equal, hence N_2 at least is distinct from M and N_1. Pedagogic cases M and N_1 need not be distinct, although s_1 must be non-zero. For example, if M is the starting configuration, s_1 could be a repeatable dragging sequence involving one or more of points B, C and D, which shows the family of arrowhead-type examples but does not disrupt the arrowhead configuration *per se*.

$$\begin{tikzcd} & \mathbf{N_1} \\ \mathbf{M} \arrow[ur, "s_1"] \arrow[dr, "s_2"'] & \\ & \mathbf{N_2} \end{tikzcd}$$

At the end of this section, Figure 7.1 illustrates the hierarchy of minimum knowledge requirements for the four categories of the framework. It is also included as a separate sheet in the back cover of the thesis.

7.3.1 Minimum knowledge requirements for Unarticulated, Unanticipated opportunities

Robert's exploratory dragging of D1 produced a sequence of four pedagogic cases 21-17-16-19. This provided an instance of an unarticulated, unanticipated choice and use of examples and decision about sequencing. He appeared unaware of the pedagogic cases he elicited and of the dragging sequence needed to elicit these four pedagogic cases since he did not comment on either. Thus it is not necessary to recognise any dragging sequence *s* or any pedagogic case elicited N for an unarticulated, unanticipated choice and use of examples and decision about sequencing. In other words, it is only necessary to recognise that the angle at the centre theorem holds for some configuration M of D1 *a priori*, most obviously the starting configuration, and that dragging points B, C and D will produce further (unspecified) examples of the angle at the centre theorem. The minimum knowledge requirements are therefore as follows:

- knowing some configuration M of D1, e.g. the starting (arrowhead) configuration, as a case of the angle at the centre theorem. It is necessary to know M *a priori*, i.e. before dragging.
- knowing that one or more of points B, C and D can be dragged and how to drag them using the mouse i.e. technical knowledge.
- knowing that dragging points B, C and D produces mathematical variation. The nature of this variation may be unspecified i.e. dragging B, C or D causes the diagram D1 to change somehow. How it changes or precisely what causes the change, beyond dragging B, C or D, is not necessary knowledge.
- knowing that one or more of points B, C or D should be dragged and dropped in a variety of positions to convincingly demonstrate the angle at the centre theorem. However, such dragging may be unsystematic and therefore unrepeatable i.e. no routine for dragging.

7.3.2 Minimum knowledge requirements for Unarticulated, Anticipated opportunities

Robert, Anne and Edward elicited pedagogic case 3 by dragging point B as part of their apparently routine demonstration of the angle at the centre theorem using D1. These provided instances of an unarticulated, anticipated choice and use of examples. To some extent, each of them commented on the dragging sequence needed to elicit pedagogic case 3, yet none of the three case study teachers seemed aware of the case itself since they did not comment on it. Thus for an unarticulated, anticipated choice and use of examples, in addition to the pedagogic case M, it is necessary to recognise *a priori* the dragging sequence *s*, so that it is anticipated i.e. routine. However, it is unnecessary to recognise any pedagogic case N that might be elicited.

Michael and Anne elicited sequences of cases 1-6-12 and 1-3-5-4 respectively as part of their apparently routine demonstration of the angle at the centre theorem using D1. These provided examples of an unarticulated, anticipated decision about sequencing. To some extent, Michael and Anne each commented on the dragging sequence needed to elicit their sequence of cases, however they did not compare it to another dragging sequence. Thus for an unarticulated, anticipated decision about sequencing it is again only necessary to recognise *a priori* the dragging sequence *s*, in addition to the pedagogic case M. Although Michael did recognise all three pedagogic cases elicited through his dragging sequence, Anne's apparent unawareness of pedagogic case 3 implies it is not necessary to recognise any pedagogic case N that might be elicited. The minimum knowledge requirements are therefore as follows:

- knowledge required for Unarticulated, Unanticipated opportunities and in addition,
- knowing a dragging sequence *s* i.e. a repeatable routine for dragging one or more of points B, C and D, which elicits some pedagogic case N distinct from M.

7.3.3 Minimum knowledge requirements for Articulated, Unanticipated opportunities

This sub-section attempts to posit minimum knowledge requirements for three possible types of articulated, unanticipated choice and use of examples. The first type is one where the choice and use of examples may be unrepeatable. The second and third are both types where the choice and use of examples is repeatable, but which derive from different knowledge requirements *a priori*. Similarly, this sub-section also attempts to posit minimum knowledge requirements for three possible types of articulated, unanticipated decisions about sequencing, the first of which may be unrepeatable; the second and third are repeatable but again derive from different knowledge requirements *a priori*.

Articulated, unanticipated choice and use of examples that may be unrepeatable

After indirect prompting, Michael elicited pedagogic case 19, the arrowhead configuration displaying the reflex angle at the centre, which he commented on as being different to the starting configuration. This provided an example of an articulated, unanticipated choice and use of examples. Thus an articulated, unanticipated choice and use of examples implies it is necessary to recognise N as a pedagogic case distinct from some M *a posteriori*, i.e. after dragging sequence *s*. Immediately after eliciting pedagogic case 19, Michael exclaimed “okay, so now I’m not quite sure how I’ve done that”. His statement implies it is not necessary to recognise any dragging sequence *s* used to elicit pedagogic case N from M, however in this case the articulated, unanticipated choice and use of examples may be unrepeatable. The minimum knowledge requirements for an articulated, unanticipated choice and use of examples, which may be unrepeatable, are therefore as follows:

- knowledge required for Unarticulated, Unanticipated opportunities and in addition,
- knowing one or more criteria for distinguishing pedagogic cases (criteria can be incomplete) e.g. ‘not an arrowhead’ or ‘angle displayed at the centre is different’.
- knowing one or more criteria for judging pedagogic benefit of introducing pedagogic case N via some dragging routine *s*.

Repeatable, articulated, unanticipated choice and use of examples (I)

After indirect prompting, Edward noted he hadn’t yet explored “the problem of what happens if you drag D all the way round there”, referring to dragging point D up towards and past point B at the top of the circle. He then appeared to elicit pedagogic case 12 unexpectedly, by dragging point D in the way he described. This also provided an instance of an articulated, unanticipated choice and use of examples. However, in this case Edward describes the dragging sequence *s a priori*, thus it is likely that the articulated, unanticipated choice and use of examples would be repeatable. In this case, the minimum knowledge requirements for a repeatable articulated, unanticipated choice and use of examples, where the dragging sequence *s* is known *a priori*, are as follows:

- knowledge for Unarticulated, Anticipated opportunities and in addition,
- knowing one or more criteria for distinguishing pedagogic cases (criteria can be incomplete) e.g. 'not an arrowhead' or 'angle displayed at the centre is different'.
- knowing one or more criteria for judging pedagogic benefit of introducing pedagogic case N via some dragging routine *s*.

Repeatable, articulated, unanticipated choice and use of examples (II)

Immediately after a prompt regarding special or extreme cases of the angle at the centre theorem, Robert said, "I'd probably want to show this [drags C so that CD is a diameter]", and elicited pedagogic case 13, representing the angle in a semi-circle theorem. This provides another type of a repeatable, articulated, unanticipated choice and use of examples. This configuration seemed familiar to Robert, yet he had apparently not considered dragging point C or D far enough to make the chord CD a diameter as part of his routine dragging of D1. Another example of this type of repeatable, articulated, unanticipated choice and use of examples is the author's own dragging to elicit the upside-down arrowhead configuration. Here, there was a clear intention to elicit a particular configuration, but the author improvised a dragging sequence to arrive at the upside-down arrowhead during her first GeoGebra interview, gradually improving the efficiency of her dragging over the course of the interviews. In these cases, knowing pedagogic case N *a priori* is a necessary requirement, however it is only necessary to recognise the dragging sequence *s a posteriori*. In this case, the minimum knowledge requirements for a repeatable articulated, unanticipated choice and use of examples, where the pedagogic case N is known *a priori*, are as follows:

- knowledge for Articulated, Unanticipated choice and use of examples that may be unrepeatable and in addition,
- knowing one further pedagogic case N distinct from M e.g. apart from the starting configuration.

Articulated, unanticipated decisions about sequencing that may be unrepeatable

Articulated decisions about sequencing involve expressing a pedagogical preference for one of two recognised sequences of cases, starting from the same pedagogic case M, as depicted in the diagram in the introduction to this section. For articulated, unanticipated decisions about sequencing, one of these sequences, involving dragging sequence s_1 to elicit pedagogic case N_1 say, is necessarily recognised *a priori*, with the other being recognised only after dragging. This gives rise to an unanticipated comparison and expression of preference for one of the two sequences of cases. Thus knowing pedagogic cases M, N_1 and dragging sequence s_1 *a priori* is a necessary requirement, and in addition it is necessary to recognise N_2 as distinct from M, N_1 *a posteriori*. Pedagogic cases M and N_1 may belong to the same family of cases i.e. they need not be distinct, although s_1 must be non-zero, otherwise the decision about sequencing collapses to a choice and use of examples. It is not necessary to recognise any dragging sequence s_2, used to elicit pedagogic case N_2 from M, however in this case the articulated, unanticipated decision about sequencing may be unrepeatable. The minimum knowledge requirements for an articulated, unanticipated decision about sequencing, which may be unrepeatable, are therefore as follows:

- knowledge for an Articulated, Unanticipated choice and use of examples that may be unrepeatable and in addition,
- knowing a non-zero dragging sequence s_1, i.e. a repeatable routine to elicit pedagogic case N_1 from M, where M and N_1 need not be distinct.
- knowing one or more criteria for judging the pedagogic benefit of introducing pedagogic case N_2 via dragging sequence s_2 compared to introducing pedagogic case N_1 via dragging sequence s_1.

Repeatable, articulated, unanticipated decisions about sequencing (I)

Edward stated he would drag B to elicit pedagogic case 5, as a first "complication", before continuing to drag points C and D to elicit pedagogic case 13, the angle in a semi-circle, and pedagogic case 12, the convex quadrilateral configuration displaying the correct angle at the centre. This also provided an example of an articulated, unanticipated decision about sequencing: Edward expressed a preference for the sequence 1-5 over the unanticipated sequence 1-6-13-12. The sequence 1-6-13-12 was unanticipated because Edward elicited pedagogic cases 12 and 13 unexpectedly. In this instance, Edward had articulated the sequence 1-5 beforehand, elicited through his routine dragging, before going on to recognise pedagogic case 12 and 13 *a posteriori*. However, Edward had also commented on the dragging sequence to elicit pedagogic cases 12 and 13 *a priori*, as described under the sub-section *repeatable, articulated, unanticipated choice and use of examples* (I). Since Edward recognised dragging sequence s_2 *a priori*, it is likely that the articulated, unanticipated decision about sequencing would be repeatable. Thus, in addition to knowing pedagogic cases M, N_1 and dragging sequence s_1 *a priori*, knowing the dragging sequence s_2 *a priori* is a necessary requirement for a repeatable articulated, unanticipated decision about sequencing. It is also necessary to recognise N_2 as distinct from M, N_1 *a posteriori*. Hence the minimum knowledge requirements for a repeatable articulated, unanticipated choice and use of examples, where the dragging sequence s_2 is known *a priori*, are as follows:

- knowledge for an Articulated, Unanticipated decisions about sequencing that may be unrepeatable and in addition,
- knowing one further dragging sequence s_2, i.e. a repeatable routine to elicit some pedagogic case N_2 distinct from M.

Repeatable, articulated, unanticipated decisions about sequencing (II)

Robert suggested he would drag B first, hoping his students would prompt him to move B onto the 'wrong' segment, thus eliciting pedagogic case 5. He suggested he would then continue his demonstration by dragging points C and D, presumably eliciting pedagogic cases 6, 13 as he had done earlier in the interview. Thus Robert implied he would elicit the sequence of cases 1-5 before the unanticipated sequence 1-6-13. Again this provided an example of another type of repeatable, articulated, unanticipated decisions about sequencing. In this instance, Robert appeared to articulate pedagogic case 12 as well as the sequence of cases 1-5 *a priori*, before unexpectedly recognising the dragging sequence needed to elicit case 12, as described under the sub-section *repeatable, articulated, unanticipated choice and use of examples* (II). Thus, in addition to knowing pedagogic cases M, N_1 and dragging sequence s_1 *a priori*, knowing pedagogic case N_2 *a priori* is a necessary requirement for this type of repeatable articulated, unanticipated decisions about sequencing. It is also necessary to recognise dragging sequence s_2 *a posteriori*. Hence the minimum knowledge requirements for a repeatable articulated, unanticipated choice and use of examples, where the dragging sequence N_2 is known *a priori*, are as follows:

- knowledge for an Articulated, Unanticipated decisions about sequencing that may be unrepeatable and in addition,
- knowing one further pedagogic case N_2 distinct from M e.g. apart from the starting configuration.

7.3.4 Minimum knowledge requirements for Articulated, Anticipated opportunities

This sub-section summarises the posited minimum knowledge requirements for an articulated, anticipated choice and use of examples and decision about sequencing based on the minimum knowledge required for unarticulated, unanticipated opportunities.

Articulated, anticipated choice and use of examples

In their apparently routine response to using D1 to demonstrate the angle at the centre theorem, each of the case study teachers elicited and articulated two pedagogic cases beyond the starting configuration. In the interview context, some of these instances actually appeared more like an articulated, unanticipated choice of examples. However, it is possible to imagine that if the case study teachers had planned their demonstrations in advance, as they might for a lesson, then these would have been clear-cut instances of articulated, anticipated choice and use of examples. For example, Edward dragged point B to elicit pedagogic case 5 after saying:

> But now ... obviously there's an issue of what about if you move B on the other side. Before we do that though, let's see what G does.

In this statement, Edward anticipated that the dragging sequence "move B onto the other side [of C or D]" would raise "an issue" i.e. the pedagogic case 5, distinct from the starting configuration. From Edward's statement it is possible to infer that he had prior knowledge of both the pedagogic case 5 and the dragging sequence necessary to elicit it from the starting configuration. Hence this provided one of the clearest instances of articulated, anticipated choice and use of examples. Thus for an articulated, anticipated choice and use of examples, in addition to knowing the pedagogic case M, it is necessary to know both a pedagogic case N, distinct from M, and a dragging sequence *s* to elicit N from M. Hence the minimum knowledge requirements for an articulated, anticipated choice and use of examples are as follows:

- knowledge required for Unarticulated, Unanticipated opportunities and in addition,
- knowing one or more criteria for distinguishing pedagogic cases (criteria can be incomplete) e.g. 'not an arrowhead' or 'angle displayed at the centre is different'.
- knowing one further pedagogic case N distinct from M e.g. apart from the starting configuration.
- knowing a dragging sequence *s* i.e. a repeatable routine for dragging one or more of points B, C and D, which elicits some pedagogic case N distinct from M.
- knowing one or more criteria for judging pedagogic benefit of introducing pedagogic case N via some dragging routine *s*.

Articulated, anticipated decisions about sequencing

Robert articulated a decision to drag D1 to elicit the sequence of cases 1-5 before the sequence 1-6. Similarly, Edward articulated a decision to drag D1 to elicit the sequence of cases 1-1 before the sequence 1-6. In both instances, the case study teacher had articulated both sequences of cases before stating their preferential ordering. Hence these provided examples of an articulated, anticipated decisions about sequencing. Thus for articulated, anticipated decisions about sequencing knowing pedagogic cases M, N_1, and N_2 as distinct from M and dragging sequences s_1 and s_2 *a priori* are necessary requirements. Hence the minimum knowledge requirements for articulated, anticipated decisions about sequencing are as follows:

- knowledge required for Unarticulated, Unanticipated opportunities and in addition,
- knowing one or more criteria for distinguishing pedagogic cases (criteria can be incomplete) e.g. 'not an arrowhead' or 'angle displayed at the centre is different'.
- knowing one or more criteria for judging pedagogic benefit of introducing pedagogic case N via some dragging routine *s*.
- knowing a non-zero dragging sequence s_1, i.e. a repeatable routine to elicit pedagogic case N_1 from M, where M and N_1 need not be distinct.

- knowing one further dragging sequence s_2, i.e. a repeatable routine to elicit some pedagogic case N_2 distinct from M.
- knowing one further pedagogic case N_2 distinct from M e.g. apart from the starting configuration.
- knowing one or more criteria for judging the pedagogic benefit of introducing pedagogic case N_2 via dragging sequence s_2 compared to introducing pedagogic case N_1 via dragging sequence s_1.

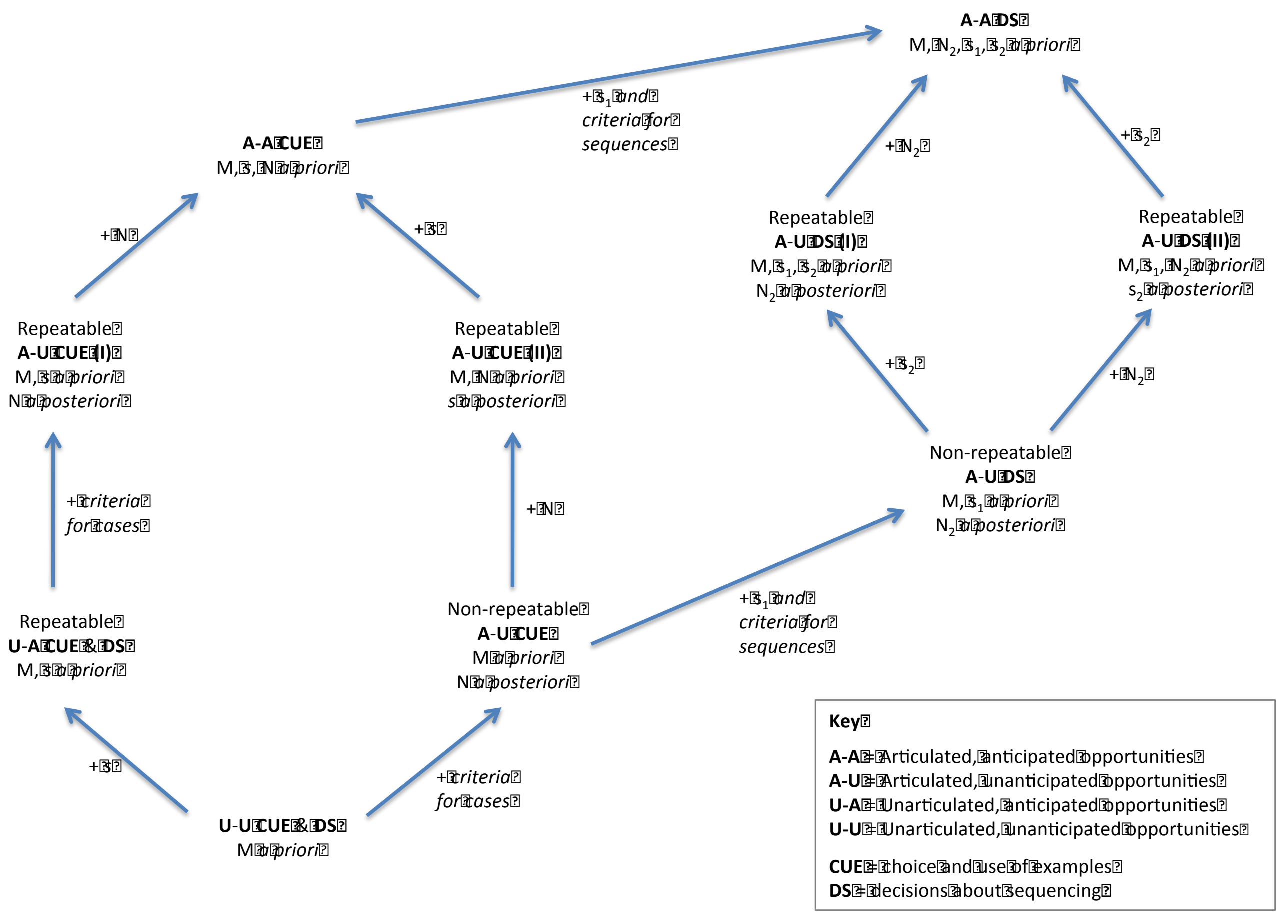


Figure 7.1 Hierarchy of minimum knowledge requirements

7.4 Explaining Michael's circle theorem case list: distributed cognition

Michael's circle theorem case list was surprising in two ways when compared to those of the other three case study teachers (see also Chapter 6). First, his circle theorem case list was the only one to indicate the pedagogically preferable sequence of cases, eliciting pedagogic case 12, the convex quadrilateral configuration displaying the 'correct' angle at the centre through dragging point C or D, before eliciting pedagogic case 5, the convex quadrilateral configuration displaying the 'wrong' angle at the centre through dragging point B. Secondly, his circle theorem case list also had the highest total number of pedagogic cases elicited by the case study teacher in the interview. That his circle theorem case list should exhibit these features was surprising because Michael was one of the least experienced case study teachers, being only in his second year of teaching. In particular, he appeared to be the least mathematically confident of the four teachers and, with his undergraduate and masters degrees in economics, was hoping to transition to becoming an economics teacher. Hence, superficially at least, Michael appeared likely to have the least mathematical knowledge for teaching circle theorems using technology, yet based on these indicators, the circle theorem case list he elicited made the most mathematical knowledge available in the interview.

The minimum knowledge requirements provide a means of explaining how a teacher with less mathematical knowledge for teaching might elicit a pedagogically preferable sequence of cases than teachers with more mathematical knowledge for teaching. Knowing a dragging routine is enough to produce a particular sequence of cases because generating and sequencing examples of the angle at the centre theorem can be delegated to the GeoGebra software. Thus a teacher simply knowing one good dragging routine might elicit a pedagogically preferable sequence of cases than another teacher who articulates a decision about sequencing, comparing two or more dragging routines, based on incomplete or otherwise flawed criteria. This seems a plausible explanation for the unarticulated, anticipated decision about sequencing Michael produced in the GeoGebra interview. Michael's routine response produced the sequence of pedagogic cases 1-6-12, through dragging points C and D to initiate a demonstration of the angle at the centre theorem. At no point did Michael drag point B to disrupt the arrowhead configuration until directly prompted to by the author. Thus he appeared to know one good dragging routine: briefly drag each of points B, C and D without disrupting the arrowhead configuration, then drag points C and D up towards B. It is also possible he followed this dragging routine based on advice from his Head of Department. On the other hand, Robert and Edward articulated a decision about sequencing, implying they recognised at least two dragging routines and based their decision on some criteria regarding the pedagogic benefit of the dragging routines. Thus they appeared to know more than Michael. They recognised the pedagogic benefit in making a connection between the angle at the centre theorem and the angles in the same segment theorem by dragging B to elicit pedagogic case 5. However, their criteria for comparison were incomplete, since they did not appreciate that making this connection, before establishing the angle at the centre theorem, might distract from the theorem itself. Hence it was possible for Michael, apparently knowing less than Robert or Edward, to elicit a pedagogically preferable sequence of cases.

All four case study teachers fulfilled the minimum knowledge requirements for an articulated, anticipated choice and use of examples, each eliciting two pedagogic cases beyond the starting configuration. Table 7.2 shows the number of pedagogic cases each case study teacher elicited in each of the four categories of the second dimension of the framework for identifying instances of distributed cognition. In particular, Michael elicited eight pedagogic cases in an articulated, unanticipated choice and use of examples compared to the other case study teachers who each elicited four. It is this category that contributes most to Michael's circle theorem case list having the highest total number of pedagogic cases elicited by the case study teacher in the interview.

Table 7.2 The number of pedagogic cases elicited in each of the four categories of the second dimension of the framework for identifying instances of distributed cognition

	UU	**UA**	**AU**	**AA**	**Total elicited**
Robert*	4	1	4	3	12
Anne	2	1	4	3	10
Edward	2	1	4	3	10
Michael	4	0	8	3	15

AA = articulated anticipated; AU = articulated unanticipated; UA = unarticulated anticipated; UU = unarticulated unanticipated.

The minimum knowledge requirements for an articulated, unanticipated choice and use of examples suggest that generating pedagogic cases in this category could be mostly delegated to the GeoGebra software e.g. for an *articulated, unanticipated choice and use of examples that may be unrepeatable* there is no need to know either a dragging sequence or a pedagogic case distinct from the starting configuration *a priori*. Hence, despite appearing to be the least mathematically confident of the four teachers, Michael was able to elicit more pedagogic cases through exploratory dragging than the other case study teachers. As a result, the mathematical knowledge made available in Michael's interview appears more effectively distributed across the GeoGebra software than in the interviews of the other case study teachers. Thus viewing cognition as distributed provides a plausible explanation for how Michael could produce a circle theorem case list with the highest total number of pedagogic cases elicited by the case study teacher in the interview.

7.5 Instances of distributed cognition from classroom observations

In this section, data from observations of the case study teachers' use of technology are used to provide two instances where cognition appears to be distributed 'in the wild' of mathematics teachers' classroom practice. The first instance exemplifies the unarticulated, anticipated category of the framework where the mathematical knowledge made available in the lesson appeared to be distributed over technology, in this case the *MyMaths* website. The second instance provides an example of an articulated, unanticipated opportunity where the mathematical knowledge made available in the lesson appeared to be distributed over a student's query in relation to a textbook question.

The first instance, where the mathematical knowledge made available in the lesson appeared to be distributed over technology, occurred in the observation of Michael's computer suite lesson. Michael had asked his year 8 pupils to work through a *MyMaths lesson* on calculating the volume of cuboids, comprising a series of slides incorporating explanations and exercises on the topic. In particular, the *MyMaths lesson* on volume included two animated slides offering a justification for the formula for the volume of a cuboid e.g. $volume = width \times length \times height$. The justification made a connection between a counting-cubes procedure for working out the volume of a cuboid and the multiplicative formula by animating the repeated addition of cuboids of unit height.

Thus the animation used a sequence of cuboids with heights increasing by an increment of one unit to make a *connection between procedures*, a code from the Connection category of the Knowledge Quartet. This connection was made available in the lesson by the decisions about sequencing and the choice and use of examples embedded in the design of the *MyMaths lesson*. The animation culminated in a 3×5×6 cuboid. Since the three dimensions of the cuboid are distinguishable by virtue of their differing lengths, the choice of this example also made it possible to demonstrate the interchangeability of the labels ‘length’, ‘width’ and ‘height’, although this was not explicitly stated on the animated slide.

In the post-observation interview, Michael suggested he thought the animation was pedagogically useful and something that was not easily done using other resources.

> It’s a nice neat little demonstration I think, that’s definitely something you wouldn’t be able to do ... in a book you wouldn’t be able to do that. I think that’s a real nice demonstration. [Mic-CS-int, 31.5.2012]

However, Michael didn’t articulate why the animation was pedagogically useful for teaching volume, beyond being visually appealing. In particular, in preparing for the lesson, it seemed unlikely Michael had considered the usefulness of the *MyMaths lesson* from the point of view of mathematics pedagogy. After the post-observation interview, Michael commented that he felt he relied on the *MyMaths* website too much, trusting the software so that he didn’t think about how to structure his explanation of mathematical concepts. During the lesson, he did not draw pupils’ attention specifically to the animated slide, although he did encourage them to work through all the slides and not just to complete the assessed exercise at the end. Before taking his pupils to the computer suite, Michael provided a worked example of calculating the volume of a cuboid on the IWB in his normal classroom for his students to copy down into their exercise books. This example was not informed by the justification provided by the *MyMaths* animated slide. Michael used a cuboid with dimensions 1×1×5 to demonstrate the volume formula. This is a poor choice and use of examples for at least two reasons. Firstly, counting cubes appears a more efficient procedure than using the volume formula in this case, hence justifying the formula is problematic. Secondly, the repetition of unit length for two dimensions of the cuboid makes it difficult to demonstrate the interchangeability of the labels ‘length’, ‘width’ and ‘height’. Michael also adopted an algebraic approach in helping pupils to find a missing length given the volume of a cuboid, rather than connecting his explanation to the concept of volume. He assumed pupils' difficulty with this type of question stemmed from an aversion to the division operation, rather than to a lack of conceptual understanding of volume. In general, Michael said he did not understand why pupils had difficulties with this topic but did later acknowledge that the vocabulary of length, width, breadth, depth, base and so on and the interchangeability of these terms could be confusing.

Michael's use of the animated slide as part of the *MyMaths lesson* provides an example of an unarticulated, anticipated choice and use of examples and decisions about sequencing. Michael planned to use the slide as part of the *MyMaths lesson* on volume and thought it was a "nice demonstration" at least in general pedagogic terms. In this sense it was an anticipated opportunity. However, Michael did not articulate the usefulness of the slide in terms of mathematics pedagogy. In particular, it seems unlikely that he had considered the pedagogic structuring provided by the *MyMaths lesson*, in terms of the choice and use of examples or decisions about sequencing embedded in the software. It also seems unlikely that Michael would have made a connection between the counting-cubes procedure and volume formula available in the lesson without the use of a resource like the *MyMaths* website. Thus Michael had a readerly response to the software: the choice and use of examples and decisions about sequencing were mainly delegated to the technology. Hence Michael's use of the *MyMaths lesson* provides an instance of distributed cognition, where the mathematical knowledge made available in the lesson appeared to be distributed mainly over the technology, likely leading to a better outcome than would have been possible without the use of a resource like the *MyMaths* website.

The second instance, where the mathematical knowledge made available in the lesson appeared to be distributed over a student in interaction with the textbook, occurred in the observation of Edward's IWB lesson. The aim of Edward's lesson was to teach his year 12 students that the graph of an inverse function $y = f^{-1}(x)$ is a reflection in the line $y = x$ of the original function $y = f(x)$. Towards the end of the lesson the students were working on an exercise in relation to this topic from the course textbook. One of Edward's students drew his attention to a textbook question where they were asked to find the inverse function of $f(x) = 4 - x$. The student thought his answer must be wrong because he had $f^{-1}(x) = 4 - x$ and couldn't understand how the function could be an inverse of itself. The student's query prompted a sudden insight for Edward into the value of this question in terms of mathematics pedagogy. He realised that the line $y = 4 - x$ is perpendicular to the line $y = x$, thus the reflection of $y = f(x)$ in the line $y = x$

simply maps the function back onto itself. As a result, the graph of the inverse function must also be $y = 4 - x$, explaining why function is self-inverse. This question was pedagogically useful because it provided an opportunity to consider why this function was self-inverse, and thus by extension what other functions might be self-inverse, whilst linking to the main aim of the lesson. In particular, the choice of example provided by the textbook question presents an opportunity to *make connections between concepts* (Knowledge Quartet, Connection), e.g. connecting the global property of having reflective symmetry in the line $y = x$ with the concept of a self-inverse function. This insight caused Edward to *deviate from his lesson agenda* (Knowledge Quartet, Contingency), drawing the attention of the whole class to this example and replacing the planned example of finding the inverse function of $y = x^2 - 3$.

Edward's response to the student's query, connecting the global property of having reflective symmetry in the line $y = x$ with the concept of a self-inverse function, provides an example of an articulated, unanticipated choice and use of examples. Edward articulated the pedagogic value of the example in the lesson, through his exposition of the example, and also later in the post-observation interview (see quotes below). The choice and use of examples was unanticipated because, although he had chosen the exercise from the textbook to some extent, he had no pedagogic expectations of the questions. For example, describing his planning of the lesson in the post-observation interview Edward said:

> No no, I ignored the textbook completely. [I: Right okay.] Did what I want to do, and then went 'Oh no the textbook seems to ask random questions'... [Ed-IWB-int, 20.6.2012]

His use of the word "random" to describe the textbook questions implies he assumed there was no intentional pedagogical structuring in the exercise i.e. no deliberate choice and use of examples or decisions about sequencing embedded in the design of the exercise. Hence he did not expect the questions to be of any particular pedagogic value. In addition, Edward acknowledged during the interview that he had not looked at the questions beforehand when planning the lesson and wouldn't have thought of this "learning objective" otherwise. He still argued that the textbook sometimes appeared "random" i.e. without pedagogic structure, but that this example was an exception.

> E: It was a question from the book, but I hadn't looked at it before and I was stuck for a bit, it's like is that right or wrong ... oh that means this [inaudible] realised it was self-inverse and reflective.

I: Yeah, okay.

E: Which could have been a really good learning objective I just hadn't thought of it. So that's another example of the book coming up with points that I wouldn't have thought of. So it's not all bad, the questions in the book, they are a bit random sometimes. In fact that wasn't, that was quite a good question to ask actually. [Ed-IWB-int, 20.6.2012]

The student's query in relation to the correctness of his answer, given that it appeared to be the same as the original question, appeared instrumental in drawing Edward's attention to the pedagogic value of this particular example. Without this intervention, it seems unlikely that Edward would have recognised the pedagogic value of the question, hence the mathematical knowledge made available in the lesson appeared to be distributed over a student in interaction with the textbook. Thus Edward's response to the student's query provides an instance of distributed cognition, where the mathematical knowledge made available in the lesson appeared to be distributed over a student in interaction with the textbook, likely leading to a better outcome than would otherwise have been possible.

7.6 Summary

In this chapter, the research focus shifted from individual teachers' own knowledge to how this knowledge is involved in the participatory relationship with technology (Remillard, 2005). Hutchins' (1995) view of distributed cognition was used as a framework for understanding how individual teachers' knowledge is involved in interacting with technology to produce the mathematical knowledge made available in the classroom. In particular, specifying minimum knowledge requirements provided a means of detailing how and to what extent knowledge is distributed across teacher and technology and hence describing the participatory relationship (Remillard, 2005).

In section 7.1, Hutchins' (1995) mode of argument was applied to the case study teachers' suggestions of how they would use the GeoGebra diagram D1 to demonstrate the angle at the centre theorem to pupils. This involved mapping the elements of his framework to the participants, hypothetical or otherwise, in the GeoGebra interviews. Mapping the elements provided a means of contextualising Hutchins' view of distributed cognition in relation to this research project in particular and mathematics education research more generally.

A framework for identifying instances of distributed cognition, where the mathematical knowledge made available was more or less distributed across the technology, was developed and exemplified from an analysis of the circle theorem case lists presented in Chapter 6. The framework differentiated between the case study teachers' unarticulated and articulated choice and use of examples and decisions about sequencing. Unarticulated and articulated choice and use of examples and decisions about sequencing were identified respectively with a readerly and writerly response to D1. Identifying the case study teachers' choice and use of examples and decisions about sequencing with a readerly or writerly response was a means of indicating where the mathematical knowledge made available was more or less distributed across the technology. In addition, indicating where the mathematical knowledge made available was more or less distributed across the technology allowed the framework to expand the Knowledge Quartet, unravelling the tension bound up in the code *adherence to textbook* identified in Chapter 6.

The development of the framework for identifying instances of distributed cognition led to the specification of minimum knowledge requirements necessary for the case study teachers to produce the circle theorem case lists depicted in Chapter 6. Specifying minimum knowledge requirements provided an indication of how and to what extent the case study teacher's mathematical knowledge for teaching was distributed across the software, the author and hypothetical teaching-colleagues. More generally, the specification of minimum knowledge requirements potentially provides a theoretical means of indicating how and to what teachers' mathematical knowledge for teaching is distributed across technology and hence a means of describing the participatory relationship between teacher and technology (Remillard, 2005). The minimum knowledge requirements also provided a means of explaining why case study teacher Michael was able to produce a better circle theorem case list in pedagogical terms than the other more experienced and more mathematically confident case study teachers. This provides convincing evidence in support of a distributed view of cognition. In particular, it suggests that an individual teacher 'having' more knowledge does not necessarily equate to an improvement in the mathematical knowledge made available - at least in the interview setting. It seems reasonable to speculate that this might also generalise to a classroom setting. The last section of this chapter, data from classroom observations of the case study teachers' use of technology were used to provide brief indications of how the framework might be exemplified in mathematics teachers' classroom practice.

Chapter 8 - Mathematical knowledge for teaching using technology

The broader aim of this study was to develop a deeper understanding of both how and why mathematics teachers use technology in their classroom practice. This final chapter presents the findings of the study and the contribution this study makes to research in the fields of teacher knowledge and technology in mathematics education. Following this, the generalisability of the findings and limitations of the study are discussed. This chapter concludes by considering the implications of this study for future research, policy and practice.

8.1 Introduction

The motivation for this study was to develop a deeper understanding of teachers' classroom practice using digital technology. This led to the progressive focusing of the PhD research project on teachers' mathematical knowledge for teaching using technology. 'Technology' is defined broadly as any artefact (physical or virtual) that has been designed for use or has been appropriated for use in teaching mathematics. Similarly, 'digital technologies' is used to indicate a digital artefact (physical or virtual) that has been designed for use or has been appropriated for use in teaching mathematics. This definition of digital technologies includes software and hardware that are not obviously 'mathematical tools'. For example, teacher-centred digital technologies such as presentation-oriented software (e.g. PowerPoint, IWB software and the *MyMaths* website) are included, but non-digital technologies such as textbooks are excluded.

The first section of Chapter 1 argued that mathematics education research has tended to focus narrowly on digital technologies such as those termed *cognitive technological tools* (Zbiek et al., 2007) or mathematical analysis software (Pierce & Stacey, 2010). It was argued this has led to the perception of a deficit in teachers' use of digital technologies. The apparent deficit exists in two senses. Firstly, mathematics teachers appear to make only occasional use of digital technologies in their teaching. Secondly, when teachers *do* make use of digital technologies, the potential of these technologies to enhance pupils' mathematical experience in the classroom is rarely realised. Adopting a wider definition of digital technologies, such as the one used in this study, suggests the apparent deficit in teachers' frequency of use may be over-estimated. Teachers do integrate some types of digital technology into their classroom practice. However, these types of digital technology are commonly assumed to be teacher-centred (rather than student-centred) and to maintain or even encourage existing 'traditional' pedagogies.

Remillard's (2005) perspective provided a reminder that although the constraints and affordances inherent in digital technologies may help to shape its end use in the classroom, inevitably, teachers as end-users will also work to shape the technology. Thus connectionist-oriented teachers may work to shape digital technologies commonly assumed to be teacher-centred, using them in ways that conform to their own more student-centred pedagogy. Chapter 1 hypothesised that, if this is the case, then the apparent deficit in teachers' use of digital technologies in terms of the extent to which their potential is realised may also be over-estimated.

In addition, Chapter 1 argued there is an imperative for research on how technology use is associated with a connectionist orientation. This imperative, alongside this study's focus on teacher knowledge in relation to using technology to teach mathematics, gave rise to the two main research questions, RQ1 and RQ2, and two subsidiary questions, RQ2a and RQ2b, listed below. Since this study has defined a connectionist orientation mainly in terms of knowledge, contrasting connectionist teachers' with transmissionist teachers' use of technology provided a means of making visible individual teachers' *mathematical knowledge for teaching using technology*.

RQ1 *How is a connectionist orientation towards teaching mathematics associated with teachers' frequency of use of ICT, their orientation towards ICT and their pedagogic practices involving ICT?*

RQ2 *What is the nature and content of teachers' mathematical knowledge for teaching using technology, as represented by the central TPACK construct?*

RQ2a *How does a conceptualisation of teachers' mathematical knowledge for teaching using technology suggest ways in which such knowledge could be measured?*

RQ2b *To what extent is the mathematical knowledge made available through a teachers' interaction with technology distributed across the individual teacher and the technology?*

This study adopted a mixed-methods approach towards investigating individual teachers' knowledge and how it is involved in interacting with technology to produce the mathematical knowledge made available in the classroom. A largely quantitative approach was taken to addressing Research Question 1, using survey data to explore associations between a connectionist orientation and ICT use, in order to be able to generalise more effectively to the population of English secondary school teachers. Addressing Research Question 2 and the subsidiary questions 2a and 2b, a qualitative approach was judged necessary to investigate individual teachers' knowledge and how it is involved in the participatory relationship (Remillard, 2005) between teacher and technology. Qualitative data collection involved observing lessons where ICT was being used, followed by a post-observation interview, and a semi-structured interview or 'think aloud' based around manipulating a GeoGebra file on circle theorems.

8.2 Findings

RQ1 *How is a connectionist orientation towards teaching mathematics associated with teachers' frequency of use of ICT, their orientation towards ICT and their pedagogic practices involving ICT?*

The findings in Chapter 4 suggest that moving research on teachers' use of technology in mathematics education away from a narrow focus on cognitive tools (Zbiek et al., 2007) or mathematics analysis software (Pierce & Stacey, 2010) to include teacher-centred digital technologies such as presentation-oriented software (e.g. PowerPoint, IWB software and the *MyMaths* website) may be productive in terms of developing a deeper understanding of both how and why mathematics teachers use digital technologies in their classroom practice. In Chapter 4, a key finding was that frequent use of software commonly assumed to maintain and even encourage existing transmission-oriented pedagogies was surprisingly associated with a more connectionist orientation. Whilst mathematical analysis software is associated with making connections, for example between multiple representations (Kaput, 1992), this finding seems to suggest that 'teacher-centred' digital technologies might also be important in supporting connectionist-oriented practices. This supports the suggestion that the apparent deficit in teachers' frequency of use may be over-estimated. In particular, this finding suggests that if connectionist teachers do adapt 'teacher-centred' digital technologies in ways that support or conform to their own more student-centred pedagogy, the apparent deficit in teachers' use of digital technologies in terms of the extent to which their potential is realised may indeed be over-estimated.

In addition, Chapter 4 found that whilst there are pedagogic practices involving ICT associated with a connectionist orientation, items on pedagogic practices involving ICT that were designed to be 'teacher-centred' appear to have no association either way. These 'teacher-centred' practices involving ICT may instead be construed as 'dominant' practices, in that they are also the most frequently occurring across all teachers. Case study data supports the survey finding of dominant practices, suggesting that, superficially at least, there is little difference between connectionist and transmission teachers use of ICT. This points to a need for further research focussing on how and to what extent connectionist teachers adapt 'teacher-centred' digital technologies in ways that support or conform to their own more student-centred pedagogy – see Section 8.6 on implications of this study. Chapter 4 also provided indications that even these dominant practices make significant demands on teachers' knowledge.

RQ2 *What is the nature and content of teachers' mathematical knowledge for teaching using technology, as represented by the central TPACK construct?*

A key contribution this study makes is developing an understanding of the nature and content of individual teachers' mathematical knowledge for teaching using technology. In Chapter 1, borrowing from Shulman (1986, p.13), mathematical knowledge for teaching using technology was assumed not only to be a matter of knowing how – being competent in teaching mathematics using technology - but also of knowing what and why. In other words, mathematical knowledge for teaching using technology, as defined in this study, is when know-how or knowledge-in-action is underpinned by and coincides with the teacher's articulated knowledge. This was depicted in Figure 1.1 as the intersection of individual teachers' articulated knowledge and their knowledge-in-action. Contrasting the two transmissionist with the two connectionist case study teachers' use of technology provided a methodological means of making visible individual teachers' mathematical knowledge for teaching using technology. The literature review in Chapter 2 makes a contribution in identifying mathematical knowledge for teaching using technology as an area that has been under-researched. More specifically, the literature review found that research on technology in mathematics education has paid relatively little attention to teachers' knowledge of specific mathematical concepts in relation to technology. By comparison, research on teacher knowledge in the field of mathematics education is relatively well conceptualised, although limited in terms of its curriculum coverage. For example, Ma's (1999) research focused on arithmetic operations and Askew et al (1997) focused on numeracy. In addition, research on teacher knowledge in the field of mathematics education has rarely considered teachers' mathematical knowledge for teaching in the context of technology use. Hence the literature review concluded that the nature and content of teachers' mathematical knowledge for teaching using technology remained an unresolved question. In this respect, the TPACK framework and the central TPACK construct in particular were useful, for the purposes of this study, for concentrating attention on teacher knowledge in relation to technology, identifying mathematical knowledge for teaching using technology as a focus for research (see also Chapters 1 and 2). In a similar sense, Shulman's (1986) concept of PCK has been highly productive, stimulating research focused on the nature and content of teacher knowledge (see Chapter 2). Chapter 5 underlines the significance of mathematical knowledge for teaching using technology, showing that while a positive stance towards technology might be necessary, it is not sufficient.

The main contribution of Chapter 5 was to identify the nature of mathematical knowledge for teaching using technology with the notion of situated abstraction (Noss & Hoyles, 1996; & Kent, 2004; & Pozzi, 2002). This is indicated in Figure 8.1, by showing revisions to the theoretical framework used in this study. This notion, alongside Adler's (1999; 2001) concepts of the dilemma of transparency and visibility/invisibility, was useful in indicating the nature of mathematical knowledge for teaching using technology as abstract, mathematical knowledge and yet simultaneously as mathematical knowledge *situated* in the context of teaching using technology. In particular, the notion of situated abstraction was useful in allaying the concern that seeing mathematical knowledge for teaching using technology as situated should not be taken to mean that each piece of technology requires teachers to learn an entirely new and distinct domain of mathematical knowledge for teaching. Thus a teacher with mathematical knowledge for teaching using a static paper-and-pencil environment should find it easier to develop mathematical knowledge for teaching using GeoGebra and other technologies. Further, for the purposes of this study, Adler's concepts, linked to Lave and Wenger's (1998) metaphor of a window, help to explain how mathematical knowledge for teaching using technology can be simultaneously situated and abstract knowledge. Here, situated indicates that the context of teaching using technology provides a frame of reference through which teachers' view 'abstract' mathematical knowledge. The weak theorisation of the TPACK framework (Graham, 2011; Ruthven, 2014; Voogt et al., 2012) provided a key affordance in identifying the nature of mathematical knowledge for teaching using technology with the notion of situated abstraction. By allowing the user to move between an integrative and transformative perspective, in particular in relation to the central TPACK construct, the framework helped to balance the dual visibility and invisibility of technology in the practice of teaching school mathematics.

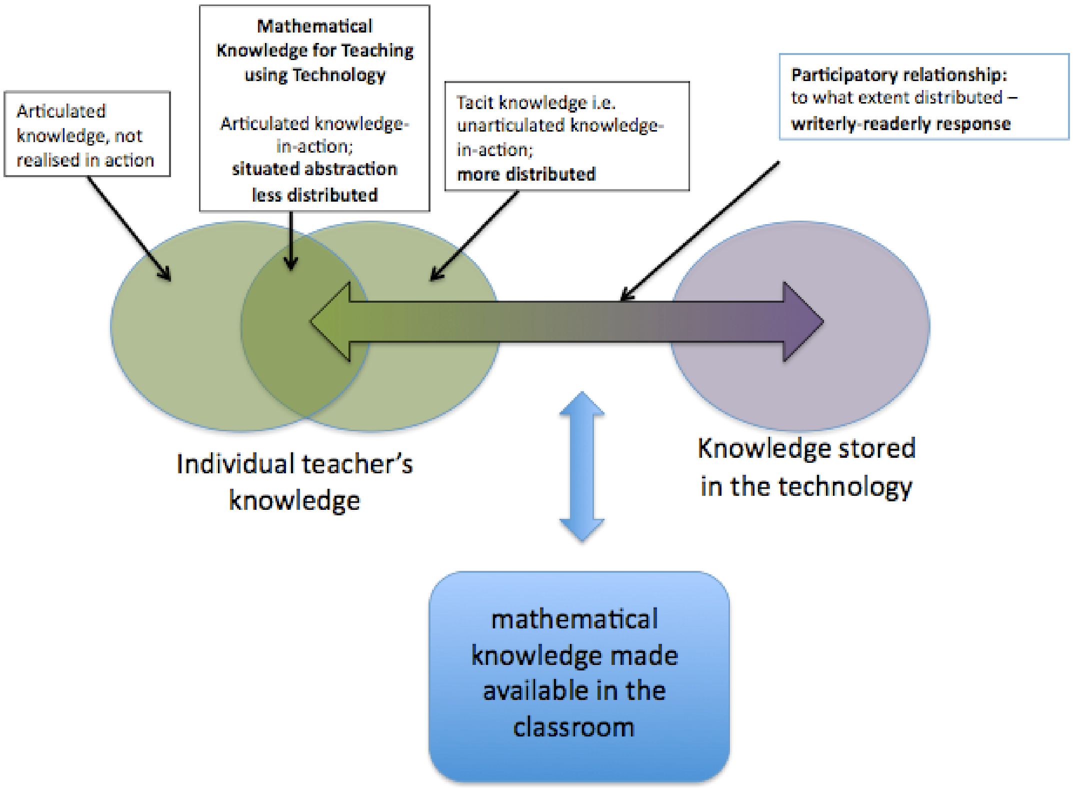


Figure 8.1 Revised framework

The analysis in Chapter 5 began to investigate the content of mathematical knowledge for teaching using technology, indicating that such knowledge could consist of a repertoire of articulated strategies for using the constraints and affordances (Greeno, 1998) of the technology for the purposes of teaching mathematics. Whilst Chapter 5 provided indications of content, Chapter 6 developed this investigation further by providing a fine-grained analysis, conceptualising the content of mathematical knowledge for teaching using technology in relation to the topic of circle theorems. The Knowledge Quartet provided a post-hoc justification for the conceptualisation of knowledge, presented in section 6.2, that forms a basis for informing and justifying a teacher's *choice and use of examples* and *decisions about sequencing*. This post-hoc justification alongside Mason and Watson's (2005; 2006) research on example spaces suggested that the approach taken in Chapter 6, i.e. mapping out a complete set of pedagogic cases and identifying a preferred pedagogical sequencing, may provide a means for conceptualising mathematical knowledge for teaching using technology for other areas of the mathematics curriculum. In addition, the analysis in Chapter 6 identifies a tension in relation to technology within the Knowledge Quartet, bound up in the code *adherence to textbook* – this tension was unravelled later in Chapter 7.

The main contribution of Chapter 6 is the demonstration of the highly complex nature of mathematical knowledge for teaching using technology, through the conceptualisation of such knowledge in relation to the topic of circle theorems. This high level of complexity provides an indication of why carrying out even the dominant practices discussed in Chapter 4, that do not go beyond enhancing general pedagogic aspirations, might prove problematic in the classroom. Reconstructing the tabular list of pedagogic cases, depicted in Figure 6.10, in practice would likely overload teachers' working memory. In addition, the high level of complexity makes it seem unrealistic for all teachers to acquire such knowledge, especially if a similar level of complex knowledge is needed for all aspects of the curriculum. Chapter 7 ameliorates this situation, providing indications of how a distributed view of cognition might offer potential strategies for facilitating teacher interaction with technology. The minimum knowledge requirements, specified in Chapter 7, suggest that knowing a dragging routine is enough to produce a particular sequence of cases. This is because generating and sequencing examples of the angle at the centre theorem can be delegated to the GeoGebra software. Thus a teacher simply knowing one good dragging routine might elicit a pedagogically preferable sequence of cases than another teacher who articulates a decision about sequencing, comparing two or more dragging routines, based on incomplete or otherwise flawed criteria. In addition, since a dragging routine brings pedagogic cases to light in a particular order, this could alleviate the demands placed on teachers' working memory in terms of recognising pedagogic cases from the tabular list depicted in Figure 6.10.

RQ2a *How does a conceptualisation of teachers' mathematical knowledge for teaching using technology suggest ways in which such knowledge could be measured?*

The conceptualisation of mathematical knowledge for teaching using technology in relation to the topic of circle theorems, presented in Chapter 6, is the type of conceptualisation that is currently under-developed in existing measures of mathematical knowledge for teaching (e.g. Baumert et al., 2010; Hill et al., 2005; Tatto et al., 2012) as argued in Chapter 2. The original intention of this study was to develop test items for measuring mathematical knowledge for teaching using technology, for example, based upon the GeoGebra file used in the semi-structured interviews. Research Question 2a was not addressed in as much depth as the other research questions due to the unexpected richness of data arising from the GeoGebra interviews and, in particular, the high complexity of conceptualising mathematical knowledge for teaching using technology in relation to the topic of circle theorems. Nevertheless, the conceptualisation presented in section 6.2 provided a basis for suggesting ways in which such knowledge could be measured. In particular, the circle theorem case lists provided a means of summarising the mathematical knowledge made available through the case study teachers' participation with a diagram in GeoGebra. The development of the circle theorem case list suggests teachers' mathematical knowledge for teaching using technology in relation to the topic of circle theorems may be summarised as a two-dimensional measure, with one dimension providing an indicator of teachers' *choice and use of examples* and the other dimension providing an indicator of their *decisions about sequencing*. The total number of pedagogic cases the case study teacher elicited in the interview provides a numerical summary as one possible indicator of teachers' choice and use of examples. Another possible numerical indicator of teachers' choice and use of examples is the percentage of explicitly recognised pedagogic cases that teachers suggested they would show their pupils. Chapter 6 conjectured that this latter indicator might show some relationship to transmissionist measure. However, these post-hoc inferences will require further research.

RQ2b *To what extent is the mathematical knowledge made available through a teachers' interaction with technology distributed across the individual teacher and the technology?*

The literature review in Chapter 2 noted that research on teacher knowledge informed by views of cognition as *distributed* (Hutchins, 1995) across persons and technology remains underdeveloped (Putnam & Borko, 1997). In particular, the extent to which the mathematical knowledge made available through a teachers' interaction with technology distributed across the individual teacher and the technology remains unclear. In Chapter 7, the main contribution was to describe the nature of the participatory relationship (Remillard, 2005) between teacher and technology in terms of a readerly/writerly response. Designating a teachers' response as readerly or writerly is not intended to as a normative judgement. Instead, a readerly or writerly response was a means of indicating where the mathematical knowledge made available was more or less distributed across the technology – see Figure 8.1. In Chapter 7, a framework for identifying instances of distributed cognition was developed and exemplified from an analysis of the circle theorem case lists presented in Chapter 6. The circle theorem case lists provided a summary of the mathematical knowledge made available through the case study teachers' participation with a diagram in GeoGebra, a type of mathematical analysis software. The development of the framework for identifying instances of distributed cognition led to the specification of minimum knowledge requirements necessary for the case study teachers to produce the circle theorem case lists depicted in Chapter 6. More generally, the specification of minimum knowledge requirements potentially provides a theoretical means of indicating how and to what extent teachers' mathematical knowledge for teaching is distributed across technology and hence a means of describing the participatory relationship between teacher and technology (Remillard, 2005). In addition, indicating where the mathematical knowledge made available was more or less distributed across the technology allowed the framework to expand the Knowledge Quartet, unravelling the tension bound up in the code *adherence to textbook* identified in Chapter 6.

8.3 Developing a deeper understanding of how and why mathematics teachers use technology in practice

Bringing the analysis together, these findings suggest that focussing on teachers' *participation with technology* (Remillard, 2005) in terms of a *readerly* or *writerly* response, in relation to the broader definition of technology used in this study, might be more productive than focusing narrowly on teachers' integration of the types of software valorised by the maths education research community (e.g. Pierce & Stacey, 2010; Zbiek et al., 2007). This is not to say that the type of technology is unimportant, but that the mathematical knowledge made available by the teacher in interaction with technology is critical. In particular, broadening the focus to include teacher-centred technologies (e.g. PowerPoint, IWB software and the *MyMaths* website) might reduce the deficit view of teachers' technology use. Hutchins (1995, p. 172) states that he is careful not to define a class of designed external tools for thinking, such as cognitive artifacts, since a distributed view of cognition suggests any artefact (used for teaching mathematics) can to some extent be viewed as 'having' cognitive attributes. Thus a distributed view of cognition should be applicable to the broader definition of digital technology adopted in this study. In section 7.6, data from classroom observations of the case study teachers' use of technology were used to provide brief indications of how the framework for identifying instances of distributed cognition, developed in Chapter 7, might be exemplified in mathematics teachers' classroom practice. In particular, these brief indications provide examples of how and to what extent mathematical knowledge made available through a teacher's interaction with technology might be distributed across 'teacher-centred' digital technologies and non-digital technologies, such as textbooks. For example, Michael's participation with the *MyMaths* website in his lesson on volume provided an instance of a readerly response to a 'teacher-centred' technology, where an explanation for the formula for the volume of cuboid appeared more distributed across the technology than the teacher. In Edward's lesson, his insight in relation to a student's query concerning a textbook question provided an instance of a writerly response, where the identification of self-inverse functions appeared somewhat more distributed across the teacher than the technology and the student. Further indications are needed of how the framework for identifying instances of distributed cognition might be exemplified in mathematics teachers' classroom practice. In particular, examples are needed of how and to what extent mathematical knowledge made available through *connectionist* teachers' interaction with 'teacher-centred' digital technologies might be distributed across the technology and teacher.

8.4 The generalisability of this study

This section addresses the generalisability of findings from this study. In both the quantitative and qualitative phases of this study, the sample of teachers upon which the findings were based represented a theoretically purposive choice. In this sense, in both phases of data collection, case selection aimed for an atypical sample rather than a representative sample of the underlying population of teachers. For this reason, the findings from this study cannot be generalised in a simplistic way to a wider population of teachers without further empirical validation. However, as critical cases, the samples in both phases of data collection were central to building a compelling argument for the findings set out in section 8.2. This section discusses the generalisability of claims made in relation to the quantitative and qualitative phases of this study in more detail in the following paragraphs.

The findings based on the survey data and reported in Chapter 4 in relation to associations between a connectionist orientation and teachers' frequency of use of ICT, their orientation towards ICT and their pedagogic practices involving ICT are not representative of teachers as a whole. The survey sample was purposively directed to ensure sufficient representation of connectionist teachers in the sample. The study did not aim for a statistically representative sample, which in any case was not feasible within the resources of the project. In particular, as stated in Chapter 3, the survey sample is likely to be biased towards teachers who are relatively well-disposed towards ICT or those wishing to be seen as frequent users of ICT. However, this bias may be seen as a strength of case selection in that it serves to underline the surprising nature of the association between frequent use of teacher-centred technologies and a more connectionist orientation, making this finding worthy of further investigation. Hence, there is relatively strong evidence supporting the surprising finding that frequent use of software commonly assumed to maintain and even encourage existing transmission-oriented pedagogies was associated with a more connectionist orientation.

The four case study teachers were selected as critical cases, hence this study aimed for analytic generalisability (Mitchell, 1984). Indeed, the strength of this sample lies in the purposive choice of cases: contrasting connectionist and transmissionist teachers' use of technology was a means of making visible individual teachers' mathematical knowledge for teaching using technology. In Chapter 5, the finding that the nature of mathematical knowledge for teaching using technology can be described as a situated abstraction provides a theoretical insight. Similarly, in Chapter 7, describing the participatory relationship between teacher and technology in terms of a readerly/writerly response (i.e. as more or less distributed) and the specification of minimum knowledge requirements provide theoretical insights into mathematical knowledge for teaching using technology. The generalisability as well as the validity of such insights rests on the extent to which they are useful (Hodgen, 2003) in understanding mathematical knowledge for teaching using technology in contexts beyond this study. In other words, these theoretical insights require further empirical testing in other technological contexts, with wider samples of teachers, in different curriculum areas, in cultural contexts outside England and so on to determine the extent of their generalisability. In Chapter 6, the high complexity of the conceptualisation of mathematical knowledge for teaching using technology in relation to the topic of circle theorems seems likely to generalise beyond the four case study teachers, although another sample of teachers might experience more or less difficulty in managing the large number of pedagogic cases. The high level of complexity may be partly related to the dynamic nature of the GeoGebra software. In addition, further research would be required to test whether the centrality of the Knowledge Quartet codes *choice and use of examples* and *decisions about sequencing* generalises to other technological contexts and curricula contexts. The extent to which findings in Chapter 6 might generalise beyond the English cultural context is also a matter for further research.

8.5 Limitations of this study

The focus of this study was on individual teachers' knowledge and how it is involved in interacting with technology to produce the mathematical knowledge made available in the classroom. This study also chose to focus on the geometrical topic of circle theorems, since research on mathematical knowledge for teaching has tended to focus elsewhere, for example for example, on arithmetic operations (Ma, 1999) and on numeracy (Askew et al., 1997) – see Chapter 2. In addition, this study focused on a particular piece of dynamic geometry software: the semi-structured interviews with case study teachers were based around a file designed in GeoGebra. This provided a familiar yet challenging technological context for the case teachers, enabling an investigation of the teachers' mathematical knowledge for teaching using technology. As noted in Chapter 1, no study can take all factors into account, thus all studies are necessarily limited in scope to a greater or lesser extent. This study chose not to focus on other factors which Remillard (2005) identified as important in affecting the participatory relationship between teacher and technology, such as other individual characteristics of teachers; characteristics of technology; students; and contextual features (e.g. school and departmental culture and the level of teacher support). In addition, this study is limited in terms of the technological context, the curricula topic and the English cultural context. Limiting the scope of the study in this way enabled a focus on individual teachers' knowledge and how it is involved in interacting with technology to produce the mathematical knowledge made available in the classroom.

Whilst the study did not focus on other factors affecting the participatory relationship between teacher and technology, these factors were acknowledged and where possible were taken into account. For example, the survey attempted to take into account *contextual features* (Stein et al., 2007) that might influence the participatory relationship between teachers and technology by including items on access to hardware/software and school/departmental level factors effecting ICT use. In addition, contextual features were taken into account, specifically the *working environment* (Ruthven, 2009), by dividing questions between using software in a whole-class context with an IWB and using ICT in a computer suite where students have direct access to the software. Further, the survey sample was purposively directed at school level, enabling the collection of data on *contextual features* (Stein et al., 2007) such as local school or departmental level factors effecting ICT use. For the qualitative phase of the study, the level of school support was used as one dimension of variation upon which to base the selection of the four case study teachers. During lesson observations, an effort was made to note students' comments to which the teacher responded, particularly in moments of contingency, which appeared to make demands on the teacher's mathematical knowledge for teaching using technology. Similarly, in Chapter 7, in mapping the elements of Hutchins' view of distributed cognition, the humans involved in the GeoGebra interviews were acknowledged as the author, the case study teacher, the case study teacher's students and the case study teacher's departmental colleagues – although of course the students and the case study teacher's departmental colleagues were only hypothetically involved. To an extent, the lesson observations provided opportunities to investigate mathematical knowledge for teaching using technology with a wider variety of technologies and curricula topics outside the context of the semi-structured GeoGebra interviews.

8.6 Further research and implications for policy and practice

This section sets out suggestions for future research and tentative implications for policy and practice in initial teacher education, teacher education more generally, curriculum resource design and development.

8.6.1 Research

Further research is needed to corroborate the surprising finding that frequent use of teacher-centred software was associated with a more connectionist orientation. In addition, a larger scale qualitative study should seek to explore how and to what extent connectionist teachers adapt teacher-centred digital technologies in ways that support or conform to their own more student-centred pedagogy.

Further research is also needed to examine the wider applicability of the theoretical insights provided in Chapters 5 and 7, beyond the immediate context of this study. Initially this might focus most usefully on investigating to what extent these insights extend to different curricula topics and different technological contexts, both digital and non-digital.

Similarly, further research is necessary to test whether the centrality of the Knowledge Quartet codes *choice and use of examples* and *decisions about sequencing*, in Chapter 6, generalises to other technological contexts and curricula topics. In particular, such qualitative research, aimed at developing conceptualisations of mathematical knowledge for teaching using technology, should suggest ways in which such knowledge could be measured. For example, in relation to the topic of circle theorems, the circle theorem case list provided a means of summarising teachers' knowledge and suggested indicators that could provide measures of such knowledge. A feasibility study would be needed to develop ways of automating the production of a circle theorem case list or operationalising the conceptualisation presented in Chapter 6 as a test item. A quantitative study would then be needed to test this at scale.

Chapter 6 concluded with some post-hoc inferences about possible relationships between transmissionist measure and indicators of mathematical knowledge for teaching circle theorems using technology. Specifically, further research is needed to test the hypothesis that there would be no relationship between transmissionist measure and the total number of pedagogic cases in a teacher's circle theorem case list. In addition, further research would be needed to test the hypothesis that the percentage of explicitly recognised pedagogic cases that teachers suggested they would show their pupils would be associated with transmissionist measure.

8.6.2 Policy

The quality of mathematical knowledge made available in the classroom depends in part on the design-quality of technologies for teaching mathematics. Chapter 7 suggests that well-designed technologies might support teachers in making mathematical knowledge available to their pupils in their classroom practice. For example, knowing a good routine for dragging dynamic software might allow teachers to delegate the generation and sequencing of pedagogic cases to the technology. However, this implies the quality of mathematical knowledge made available in the classroom would then rely in part on the quality of design of such technology for teaching mathematics. For example, the design of teaching technologies should pay particular attention to providing a systematic exploration of the example space of pedagogic cases i.e. pay careful attention to the choice and use of examples and decisions about sequencing to provide a mathematically coherent experience. In particular, this suggests frequent curriculum changes may be counter-productive, since the rapid re-design of textbooks in response to such changes is likely to incur a decline in their design-quality. Similarly the proliferation of web-based resources for teaching mathematics, where the design-quality is unclear, may result in a deterioration in the mathematical knowledge teachers are able to make available to their pupils in their classroom practice. On the other hand, if further research corroborates and can explain the surprising finding that frequent use of teacher-centred software was associated with a more connectionist orientation, then enhancing web-based resources such as the *MyMaths* web-site might provide a means of improving classroom practice incrementally on a wide scale.

8.6.3 Practice

An implication of this study for practice is that teacher education should cover a broader range of technology. In particular, teacher education should not be limited to a consideration of technologies such as those termed *cognitive technological tools* (Zbiek et al., 2007) or mathematical analysis software (Pierce & Stacey, 2010). Teacher education should also cover teacher-centred digital technologies such as presentation-oriented software (e.g. PowerPoint, IWB software and the *MyMaths* website).

In Chapter 5, the dilemma of transparency provides an indication of why it has been so hard to explain teachers' difficulties in integrating technology. Managing this dilemma of transparency in the practice of teaching school mathematics is perhaps a task for teacher educators in particular, as well as the mathematics education community as a whole, involving decisions about to what extent can mathematical knowledge be delegated to technology in the classroom.

In addition, this study suggests that teacher education should focus on developing teachers' routines for using technology. Case study teacher Michael's routine for dragging D1 to exemplify the angle at the centre is twice the angle at the circumference provides a possible example: his dragging routine produced an arguably better sequence of pedagogic cases than the other three case study teachers.

Finally, this study suggests that specifying minimum knowledge requirements for teaching may be possible. If further research shows this to be the case, then a focus on equipping trainees with the minimum knowledge requirements to use technology for teaching mathematics would be important in initial teacher education.

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Appendices

Appendix A. Survey instrument

Using ICT to investigate Mathematical Knowledge in Teaching

Survey of ICT Use

TEACHER QUESTIONNAIRE

The questionnaire should take 20-30 mins to complete.

Please read the information sheet overleaf before completing the questionnaire.

Nicola Bretscher
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King's College London
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Waterloo Road
London SE1 9NH

INFORMATION SHEET FOR SURVEY PARTICIPANTS

King's College Research Ethics Committee Ref: REP(EM)/10/11-74

TITLE OF STUDY: Using ICT to investigate Mathematical Knowledge in Teaching

You are being invited to participate in the main study for my PhD research project. Before you decide whether to take part, it is important for you to read the following information carefully and understand why the research is being done and what it will involve. You should only participate if you want to; choosing not to take part will not disadvantage you in any way. Please take time to read the following information carefully and discuss it with others if you wish. If you have any questions about the project, please do not hesitate to contact me using the contact details below.

What is the purpose of this study? This study aims to investigate the mathematical knowledge that teachers draw upon in their teaching of mathematics using ICT. In this project ICT is also used as a tool to explore teachers' mathematical knowledge.

Why have I been chosen to participate? Your school was chosen due to its existing contacts with King's College London. If you have received this information sheet, your head of department is willing for you to participate in this survey should you wish.

What will happen to me if I take part? If you choose to participate, please complete the attached questionnaire and seal it in the envelope provided, before returning it to your head of department – please note that completion of the questionnaire implies your consent to take part in the study. The questionnaire should take you no longer than 30 minutes to complete. You may also be invited to be a case study teacher based on your answers to the questionnaire. Should you be invited to be a case study teacher, you will receive a letter of invitation and a sheet explaining what being a case study teacher involves.

What are the possible benefits? Participation in the study may make you more aware of how you use ICT resources in your teaching and give you an opportunity to reflect on your professional practice. At the end of the study you will be able to receive information about the survey results, should you wish.

Will my personal data be kept confidential? Your confidentiality will be ensured at all times and you will not be identified in any publication. Questionnaire data will be stored anonymously for seven years. Since the questionnaire is anonymous it will not be possible to remove your individual data from the survey, following receipt of your questionnaire.

Who is organising and funding the research? The research is part of the principal investigator's PhD research at King's College London and is funded by the ESRC. This study is reviewed by the College Research Ethics Committee of King's College London.
Contact for further information: The principal investigator of this study is Nicola Bretscher, Department of Education and Professional Studies. This is part of a PhD project, supervised by Dr. Jeremy Hodgen, Department of Professional and Educational Studies. Should you have any comments or concerns resulting from your participation in the study please contact me via email: nicola.bretscher@kcl.ac.uk If this study has harmed you in any way you can contact King's College London using the details below for further advice and information: Jeremy Hodgen, email: jeremy.hodgen@kcl.ac.uk

Using ICT to investigate Mathematical Knowledge in Teaching: Survey of ICT Use

TEACHER QUESTIONNAIRE

Please note: completion of this questionnaire implies your consent to participate in this project.

A. ICT in your school

1. What hardware do you have access to for teaching maths?

- ☐ Interactive whiteboard, with a data projector
- ☐ Data projector only, linked to a computer
- ☐ Computer suite, shared with other departments
- ☐ Computer suite, dedicated to the maths department
- ☐ Class set of laptops
- ☐ Class set of graphic calculators
- ☐ Other, please specify ……………………………………………………………………………….

2. What software do you have access to for teaching maths?

- ☐ CD-ROMs
- ☐ Database (eg Microsoft Access)
- ☐ E-mail
- ☐ Graphing software (eg. Omnigraph, Autograph)
- ☐ Interactive Geometry software (eg Cabri, Geometer's Sketchpad, GeoGebra)
- ☐ Interactive whiteboard software (eg SMART tools)
- ☐ Logo
- ☐ MyMaths.co.uk website
- ☐ Other websites
- ☐ PowerPoint
- ☐ SMILE mathematics
- ☐ Spreadsheet (eg Microsoft Excel)
- ☐ Word processor (eg Microsoft Word)
- ☐ Other, please specify……………………………………………………………………………….

3. Please indicate the **strength of your agreement** with the following statements.

Write one of the numbers from 5 down to 1 against each statement, where
5 = strongly agree, **4** = agree, **3** = neither agree or disagree, **2** = disagree, **1** = strongly disagree

ICT use is a high priority in my department.	
I get support on using ICT from colleagues in my department.	
ICT resources are poorly integrated into schemes of work.	
I often have problems accessing hardware.	
Access to software is easy and reliable.	
The available software lacks relevance to the curriculum.	
The level of technical support is poor.	
I have had relevant professional development in using ICT.	

4. Please use this space to make **comments** on issues relating to **access to hardware** and/or **software**.

B. ICT use in your own mathematics teaching

1. How frequently do you use ICT for teaching mathematics, compared to other teachers in the department?

Much more frequently	More frequently	About the same	Less frequently	A lot less frequently
☐	☐	☐	☐	☐

2. Your use of hardware

a) For each type of hardware, please indicate how often you use it for teaching maths.

Please tick (☐) a box in each row. If you do not have access to the hardware tick 'Never'.

	Almost every lesson	Once a week	Once or twice a term	Annually	Never
Interactive whiteboard, with a data projector	☐	☐	☐	☐	☐
Data projector only, linked to a computer	☐	☐	☐	☐	☐
Computer suite (shared)	☐	☐	☐	☐	☐
Computer suite (maths dept only)	☐	☐	☐	☐	☐
Class set of laptops	☐	☐	☐	☐	☐
Class set of graphic calculators	☐	☐	☐	☐	☐

b) For each type of hardware, please indicate **the impact** you feel it has **on students' learning**.

Please tick (☐) a box in each row, even if you do not currently have access to the hardware.

	Substantial	Significant	Some	Very little
Interactive whiteboard, with a data projector	☐	☐	☐	☐
Data projector only, linked to a computer	☐	☐	☐	☐
Computer suite (shared)	☐	☐	☐	☐
Computer suite (maths dept only)	☐	☐	☐	☐
Class set of laptops	☐	☐	☐	☐
Class set of graphic calculators	☐	☐	☐	☐

3. **Using an interactive whiteboard or data projector in maths lessons**

a) Thinking about your use of ICT with an **interactive whiteboard or data projector** in a whole class context, please indicate how often you use each type of software in maths lessons.

If you do not have access to the software, tick 'Never'.

Please tick (□) a box in each row.	Almost every lesson	Once a week	Once or twice a term	Annually	Never
CD-ROMs	□	□	□	□	□
Database	□	□	□	□	□
Email	□	□	□	□	□
Graphing software	□	□	□	□	□
Interactive geometry software	□	□	□	□	□
Interactive whiteboard software	□	□	□	□	□
Logo	□	□	□	□	□
MyMaths.co.uk website	□	□	□	□	□
Other websites	□	□	□	□	□
PowerPoint	□	□	□	□	□
SMILE mathematics	□	□	□	□	□
Spreadsheet	□	□	□	□	□
Word processor	□	□	□	□	□

b) Again thinking about your use of ICT with an **interactive whiteboard or data projector**, please indicate the strength of your agreement with the following statements.

Write one of the numbers from 5 down to 1 against each statement, where
5 = strongly agree, 4 = agree, 3 = neither agree or disagree, 2 = disagree, 1 = strongly disagree

I am confident using ICT in lessons.	
Lessons using an interactive whiteboard/data projector take more time to prepare.	
ICT makes an important contribution to students' learning of mathematics.	
Using ICT improves student engagement in lessons.	
Students' lack of familiarity with software make lessons involving ICT difficult.	
ICT resources help students to understand mathematics.	
Classroom management is more difficult when using an interactive whiteboard/data projector.	
We cover more ground in lessons with an interactive whiteboard/data projector.	

c) Still thinking about your use of ICT with an **interactive whiteboard or data projector**, please indicate **how often** each of the following takes place in your mathematics teaching.

Write one of the numbers 5 down to 1 against each statement, where
5 = almost always, **4** = most of the time, **3** = half the time, **2** = occasionally, **1**= almost never

I use ICT for presentation purposes.	
I use ICT to generate student discussion.	
I control the software on the interactive whiteboard or data projector.	
I use ICT to follow up and explore students' ideas.	
I manage software carefully to prevent mathematical discrepancies arising.	
Students control the software on the interactive whiteboard or data projector.	
I draw attention to mathematical discrepancies in the software.	
Using ICT, I avoid students making mistakes by explaining things carefully first.	

4. Maths lessons in a computer suite or using laptops

a) Now thinking about your use of ICT **in a computer suite or with students working on laptops**, please indicate how often you use each type of software in maths lessons.

If you do not have access to the software, tick 'Never'.

Please tick (☐) a box in each row.	Almost every lesson	Once a week	Once or twice a term	Annually	Never
CD-ROMs	☐	☐	☐	☐	☐
Database	☐	☐	☐	☐	☐
Email	☐	☐	☐	☐	☐
Graphing software	☐	☐	☐	☐	☐
Interactive geometry software	☐	☐	☐	☐	☐
Interactive whiteboard software	☐	☐	☐	☐	☐
Logo	☐	☐	☐	☐	☐
MyMaths.co.uk website	☐	☐	☐	☐	☐
Other websites	☐	☐	☐	☐	☐
PowerPoint	☐	☐	☐	☐	☐
SMILE mathematics	☐	☐	☐	☐	☐
Spreadsheet	☐	☐	☐	☐	☐
Word processor	☐	☐	☐	☐	☐

b) Again thinking about your use of ICT in a **computer suite or with students working on laptops**, please indicate the strength of your agreement with the following statements.

Write one of the numbers from 5 down to 1 against each statement, where
5 = strongly agree, **4** = agree, **3** = neither agree or disagree, **2** = disagree, **1** = strongly disagree

I am confident using ICT in lessons.	
ICT lessons take more time to prepare.	
ICT makes an important contribution to students' learning of mathematics.	
Using ICT improves student engagement in lessons.	
Students' lack of familiarity with software make lessons involving ICT difficult.	
ICT resources help students to understand mathematics.	
Classroom management is more difficult in ICT lessons.	
We cover more ground in ICT lessons.	

c) Still thinking about your use of ICT in a **computer suite or with students working on laptops**, please indicate how often each of the following takes place in your mathematics teaching.

Write one of the numbers 5 down to 1 against each statement, where
5 = almost always, **4** = most of the time, **3** = half the time, **2** = occasionally, **1**= almost never

Students use ICT to practice mathematical skills.	
I encourage students to work collaboratively.	
I let students 'get a feel' for the software.	
Students explore mathematical discrepancies in the software.	
Students work on their own, consulting a neighbour from time to time.	
Students use ICT to investigate mathematical problems and concepts.	
I provide precise instructions for software use.	
I prepare software files in advance to avoid student difficulties using the software.	

5. Please use this space to make **comments on using ICT in general** in maths lessons.

C. Your own mathematics teaching in general

Please indicate how often each of the following takes place in your mathematics teaching.

Write one of the numbers 5 down to 1 against each statement, where
5 = almost always, 4 = most of the time, 3 = half the time, 2 = occasionally, 1= almost never

Students work through exercises.	
Students work on their own, consulting a neighbour from time to time.	
Students use only the methods I teach them.	
Students start with easy items and work up to harder questions.	
Students choose which questions they tackle.	
I encourage students to work more slowly.	
Students compare different methods for doing questions.	
I teach each topic from the beginning, assuming they know nothing.	
I teach the whole class at once.	
I try to cover everything in a topic.	
I draw links between topics and move back and forth between topics.	
Students work collaboratively in small groups.	
I avoid students making mistakes by explaining things carefully first.	
I tend to follow the textbook closely.	
Students discuss their ideas.	
Students work collaboratively in pairs.	
Students invent their own methods.	
Students work on substantial tasks that can be worked on at different levels.	
I tell students which questions to tackle.	
I encourage students to work more quickly.	
I go through only one method for doing each question.	
I find out which parts students already understand and don't teach those parts.	
I teach each student differently according to individual needs.	
I cover only the important ideas in a topic.	
I teach each topic separately.	
I know exactly what maths the lesson will contain.	
I jump between topics as the need arises.	

D. **About You – please tick the appropriate box in response to each question**

1. **Age:**

☐ 21 – 25 ☐ 31 – 40 ☐ 51 – 60

☐ 26 – 30 ☐ 41 – 50 ☐ 61+

2. **Gender:** ☐ Male ☐ Female

3. **Length of Service:** How many years have you been teaching?

☐ 1 year (NQT) ☐ 7-9 years ☐ over 25 years
☐ 2-3 years ☐ 10-15 years
☐ 4-6 years ☐ 16-25 years

4. **Your Position:** please tick the box which reflects your main responsibility.

☐ Head of Department
☐ 2i/c or deputy head of department
☐ Key Stage coordinator, please specify the key stage: ………
☐ Classroom teacher
☐ Other, please specify: ……………………………………………………….

5. **Training:**

Please give details of educational background in the table below, including school-level mathematics qualifications (eg GCSE, A-level or equivalent), college or university level qualifications (undergraduate degree and/or postgraduate degree etc), teacher training (eg PGCE, CertEd, Bed, GTR etc).

Details of Qualification	Date	Level Awarded

E. Case Study Participation

Please tick the box below and leave your contact details if you are willing to take part in this project as a case study participant.

☐ By ticking this box, I understand that I may be invited to be a case study teacher based on my answers to the questionnaire. I agree to be contacted in the future for this purpose.

Contact details

Email: ……………………………………………………………………

Phone number: ………………………………………………………..

Thank you for completing this questionnaire.

Appendix B. GeoGebra interview protocol

The interview is colour coded to indicate where the questions target the dyadic and triadic constructs of the TPACK framework.

Colour coding: TCK PCK TPK TPCK

[*Start recording*]
In this interview, we will discuss the scenarios in the GeoGebra files I emailed to you.

During the interview, you will find it helpful to manipulate the diagrams in the scenarios, especially to show how you might use them or demonstrate them to pupils. I'd like you to think out loud as you do so, to explain what you're doing.

First of all, here's a practice file just to help you practice thinking out loud as you manipulate a diagram. [*Open the practice item*]

Practice Item

So what is going on in this diagram? Please do drag the points and think out loud as you do so.

Circle Theorem item.

We'll discuss each diagram in turn and then I have some general questions about the diagrams. [*make sure that each diagram is discussed*]

The black points are the centres of each circle. Please do drag the red and blue points and think out loud as you do so.

<u>Prompts for Diagram 1: standard example</u>

- What is going on in this diagram?

- Show me how you might drag the diagram if you were demonstrating it to pupils.
- Are there any special cases or extreme cases you would want to show your pupils?
- Is there anywhere you would avoid dragging the diagram?

- Is dragging point B different to dragging points C and D? Why?
- What is happening when you drag point B here [drag point B to the 'wrong segment']. Should Ms Harris address this in class? How and why?
- What about dragging it like this? [drag 'upside down' – would you do this, why?]
- What about dragging it like this? [drag C and D up towards B to demonstrate the theorem works for reflex angles]
- Someone suggested adding a ray like this as a possible lead into proof [add ray going thru A and B]. What do you think about that?

Prompts for Diagram 2: non-example 1 (off-centre)

- What is going on in this diagram? Try dragging the red and blue points and think out loud as you do so.
- What happens if you drag point P to the circumference? Where else could you drag point P?

- Show me how you might drag the diagram if you were demonstrating it to pupils.
- Is there anywhere you would avoid dragging the diagram?
- It's quite tricky to position point P exactly on the circumference or at the centre. Does that matter - is it useful in some way or just a hindrance?
- Is this diagram any different to having 3 separate diagrams showing each theorem on its own?

Prompts for Diagram 3: non-example 2 (off-circumference)

- What is going on in this diagram?
- How does the angle at the edge change as you drag point R?
- Inside, outside the circle, at the centre? Can you place R so that the angle at the centre is 3 times the size of the angle at R?

- How would you use this diagram with pupils?
- Are there any special cases or extreme cases you would want to show your pupils?
- Similar to diagram 2, is it useful in any way or is it just a hindrance to have to position point R exactly on the circumference?

General prompts relating to all diagrams

- Some points are red and some are blue. Why do you think they are coloured differently? Would you discuss this with your pupils?

- The angles have been rounded to be whole numbers and sometimes there are rounding errors. How would you deal with that in a lesson – would you discuss it with pupils?

- If you were going address proof with your pupils, which of the diagrams do you think is best and why?

- Would you use the diagrams on an IWB or let pupils work directly with them on a computer (ignoring problems of booking computer rooms etc).
- if IWB, what would the kids gain mathematically and why not with computers (setting aside other issues e.g. behaviour);
- if Computers, what would they gain, is there anything you would demonstrate to them

- Which pupils/classes would you use the diagrams with and why? Would you use them differently for high and low-attaining pupils?

Appendix C. Post-observation interview schedule

Tell me about your preparation for this lesson.
- why choose ICT? what ICT? why?
- what resources did you draw on?
- how and why did you organise the classroom? Equipment? Students?
- how and why did you structure the lesson? activities? pupil interactions with the software? Your role?
- any differences compared to a non-ICT lesson?

What are your reflections on the lesson?
- refer to/ask about key events? choice of examples, representations?
- good points of the lesson?
- benefits of using ICT? (over other types of lesson)
- things to improve? What would you do differently?
- if you were doing this lesson with higher/lower attaining pupils, what would you change?

Give me specific examples of where you saw successful learning.
- what made it successful? How do you know?
- what difference did ICT make? How?

What are the key factors in making ICT use successful?
- in this lesson? in improving learning? Which topics?

- more generally? Who else uses ICT? Who supports your use? Key moment in your ICT development?

What do you perceive to be the potential pitfalls of ICT use?

When should it be switched off?
What would be required to overcome these pitfalls?

Appendix D. Coding with the TPACK framework

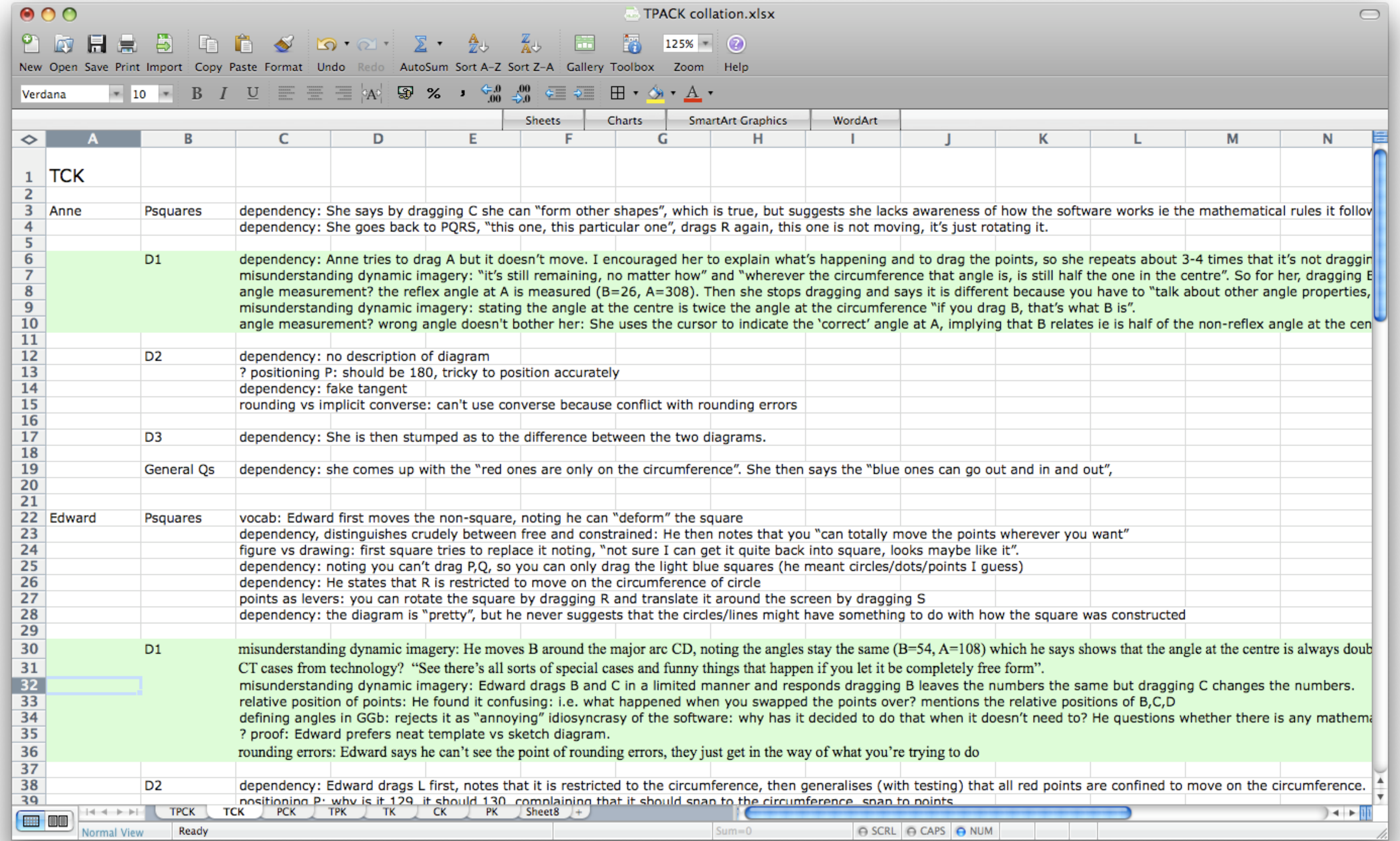
Excerpt of Robert's GeoGebra narrative coded with TPACK framework

Corresponding to Rob-GGb-int, 13.6.2012, mins 5.05-12.10

5.05 Diagram 1 (5.36) Robert quickly tries dragging A then Q. He returns to D1 and describes the diagram, noting the red points are all fixed to the circumference and are all free to move, two points (C&D) define a chord, the angles subtended etc. He states the way it currently looks it's double, then slightly displaces C and notes that due to rounding it doesn't always look that way. I think here Robert demonstrates the rounding error, rather than stumbling on it, because of the way he moves C – he's looking for badly rounded numbers. It's like he's slipping into presentation mode, presenting this to teacher-colleague? He drags G – again describing the diagram, it's there so you can change the size of the circle and see the relationship is always true. He states that he doesn't understand why the chord is there, what benefit it provides and that maybe he is missing something.	TCK: dependency TCK: rounding TPCK: pedagogic dragging strategy to demonstrate rounding PCK: pedagogic benefit (?) of chord
He's given similar diagrams to pupils before (w/o the chord and G): he thinks pupils convince themselves that the angle at the centre is double that at the circumference – he drags B in a jerky, uncoordinated way, as if mimicking a pupil. He drags C upwards, towards B, past where CD is a diameter, so the internal shape is now a convex quadrilateral, angle at the centre is reflex. Robert seems to be dragging for himself now, he speaks more softly, almost to himself "what happens if it comes this way?" - he notes it still works (100, 200). He drags C further past B "and if that comes over there?" He pauses, saying it doesn't work anymore. Of course it does work, but GGb is measuring the 'wrong' angles. Robert doesn't stop to consider why it doesn't work, but continues, coming out of his brief reverie, saying he thinks diagrams like this are almost "too easy, too obvious", "like Catchphrase" – whilst he's saying this he's slipped back into dragging it like a pupil, dragging B in a jerky way, then D, then B and D again.	TPCK: see "Catchphrase" quote below TPK: pupils' use of technology, not articulated TCK: dragging for own mathematical discovery TCK: angle measurement CK TPCK: how pupils use technology in learning maths TPK
(8.23) I ask him about how he'd demonstrate it to pupils. He asks about whether he's demonstrating the theorem to them or demonstrating how to use the diagram, so that they can explore it for themselves on a computer. I slightly obfuscate his question. Anyway, he starts to explain as if he were demonstrating the theorem. He'd have it set up with nice numbers (B=60, A=120) so that they could easily spot the relationship, near symmetrical, chord near horizontal, B near vertically above A. Before dragging B, he'd hover over the point and ask the pupils to predict what happens when he drags it to one side. Robert says he would pose the question so they didn't have the option of saying stay the same (does it get bigger or smaller?) in order to provide a bit of conflict	TPK: alternative pedagogic uses for the technology PCK: Choice of numerical & geometric example TPCK: pedagogic 'predict-then-test' dragging strategy to demonstrate invariance

for when he does drag B and the angle remains constant. He would then drag B "all the way round" the major arc CD, to see it never changes – he never gets very close to C or D. Robert then says, more softly, I can't remember what it does if I bring it through here, dragging B onto the minor arc. He says if he knew it was going to do that, he deliberately avoid it, in the hope that one of his pupils would ask. As he says this, Robert is dragging B in a peculiar way, sort of bouncing B on C and D, as if there is a sort of force field at C and D repelling his efforts to go through. When he does go "past" D, he rushes B past and onto the minor arc. When he's made diagrams like this before, he's forced B to remain on the major arc but now he wouldn't do this, he would want them to think about what's going to happen. Due to his initial chosen set up, both angles in D1 read 120 at the moment when B is on the minor arc. He wants the pupils to predict that the angle at B won't remain at 60 anymore. He adeptly adjusts some angle labels to his liking, without comment, without apparent thought, like a routine, reflex action. Robert recaps, he'd first establish that the angle at B doesn't vary in the major "segment" and then he'd start thinking about points C and D. He'd ask the pupils to make predictions about what happens to the angles when you drag D upwards? What about when angle at the centre =150 or 140, then what's B? He explains he tends to "ask before doing". He drops B too close to G and sorts out the ensuing technical difficulty with ease, returning B to a nice angle B=65, A=130. (11.16) I ask about special cases. Robert immediately drags C so that CD is a diameter, saying he would show them "Thales theorem or whatever it is", the angle in a semi-circle as a special case of the angle at the centre. Then he wonders what else could we do? He says the case where B and C meet, become the same point, dragging B onto C as he does so. He then struggles, the angle at B slowly flips between 80 and 100 (A=160), asking "what does that mean?" to himself, pauses and then "I don't know what that means..." He doesn't resolve this difficulty, and moves on, recapping that he'd bring up the angle in a semi-circle and the case where B moves on to the minor arc.	CK TPCK: pedagogic 'deliberate avoidance' dragging strategy to draw attention to variance TPCK: using technology to limit/allow pedagogic cases PCK: choice of examples TK PCK: decisions about sequencing pedagogic cases TPCK: pedagogic 'predict-then-test' dragging strategy to demonstrate variance TK PCK: choice of numerical examples PCK: choice of examples TCK: dragging for own mathematical discovery CK PCK: choice of examples

Appendix E. TPACK collation for comparison across codes and cases



TPACK collation.xlsx

	A	B	C
1	TCK		
2			
3	Anne	Psquares	dependency: She says by dragging C she can “form other shapes”, which is true, but suggests she lacks awareness of how the software works ie the mathematical rules it follov
4			dependency: She goes back to PQRS, “this one, this particular one”, drags R again, this one is not moving, it’s just rotating it.
5			
6		D1	dependency: Anne tries to drag A but it doesn’t move. I encouraged her to explain what’s happening and to drag the points, so she repeats about 3-4 times that it’s not draggin
7			misunderstanding dynamic imagery: “it’s still remaining, no matter how” and “wherever the circumference that angle is, is still half the one in the centre”. So for her, dragging E
8			angle measurement? the reflex angle at A is measured (B=26, A=308). Then she stops dragging and says it is different because you have to “talk about other angle properties,
9			misunderstanding dynamic imagery: stating the angle at the centre is twice the angle at the circumference “if you drag B, that’s what B is”.
10			angle measurement? wrong angle doesn't bother her: She uses the cursor to indicate the ‘correct’ angle at A, implying that B relates ie is half of the non-reflex angle at the cen
11			
12		D2	dependency: no description of diagram
13			? positioning P: should be 180, tricky to position accurately
14			dependency: fake tangent
15			rounding vs implicit converse: can't use converse because conflict with rounding errors
16			
17		D3	dependency: She is then stumped as to the difference between the two diagrams.
18			
19		General Qs	dependency: she comes up with the “red ones are only on the circumference”. She then says the “blue ones can go out and in and out”,
20			
21			
22	Edward	Psquares	vocab: Edward first moves the non-square, noting he can “deform” the square
23			dependency, distinguishes crudely between free and constrained: He then notes that you “can totally move the points wherever you want”
24			figure vs drawing: first square tries to replace it noting, “not sure I can get it quite back into square, looks maybe like it”.
25			dependency: noting you can’t drag P,Q, so you can only drag the light blue squares (he meant circles/dots/points I guess)
26			dependency: He states that R is restricted to move on the circumference of circle
27			points as levers: you can rotate the square by dragging R and translate it around the screen by dragging S
28			dependency: the diagram is “pretty”, but he never suggests that the circles/lines might have something to do with how the square was constructed
29			
30		D1	misunderstanding dynamic imagery: He moves B around the major arc CD, noting the angles stay the same (B=54, A=108) which he says shows that the angle at the centre is always doub
31			CT cases from technology? “See there’s all sorts of special cases and funny things that happen if you let it be completely free form”.
32			misunderstanding dynamic imagery: Edward drags B and C in a limited manner and responds dragging B leaves the numbers the same but dragging C changes the numbers.
33			relative position of points: He found it confusing: i.e. what happened when you swapped the points over? mentions the relative positions of B,C,D
34			defining angles in GGb: rejects it as “annoying” idiosyncrasy of the software: why has it decided to do that when it doesn’t need to? He questions whether there is any mathema
35			? proof: Edward prefers neat template vs sketch diagram.
36			rounding errors: Edward says he can’t see the point of rounding errors, they just get in the way of what you’re trying to do
37			
38		D2	dependency: Edward drags L first, notes that it is restricted to the circumference, then generalises (with testing) that all red points are confined to move on the circumference.
39			positioning P: why is it 129, it should 130, complaining that it should snap to the circumference, snap to points

TPCK TCK PCK TPK TK CK PK Sheet8

Normal View Ready

Appendix F. Coding with the Knowledge Quartet

Excerpt of Robert's GeoGebra narrative coded with the Knowledge Quartet

Corresponding to Rob-GGb-int, 13.6.2012, mins 5.05-12.10

Key to Knowledge Quartet codes

Foundation, Fo
AtT – adheres to textbook
AP – awareness of purpose
CoP – concentrates on procedures
IE – identifying errors
OSK – overt subject knowledge
TUP – theoretical underpinning
UT – use of terminology

Transformation, Tr
CUE – choice and use of examples
CUR – choice and use of representation
TD – teacher demonstration
UIM – use of instructional materials

Connection, Co
AC – anticipation of complexity
DS – decisions about sequencing
MCC – making connections between concepts
MCP – making connections between procedures
RCA – recognising conceptual appropriateness

Contingency, Cy
DA – deviation from agenda
RCI – responding to children's ideas
UO – use of opportunities
TI – teacher insight
RAT – responding to (un)availability of tools and resources

5.05 Diagram 1 (5.36) Robert quickly tries dragging A then Q. He returns to D1 and describes the diagram, noting the red points are all fixed to the circumference and are all free to move, two points (C&D) define a chord, the angles subtended etc. He states the way it currently looks it's double, then slightly displaces C and notes that due to rounding it doesn't always look that way. I think here Robert demonstrates the rounding error, rather than stumbling on it, because of the way he moves C – he's looking for badly rounded numbers. It's like he's slipping into presentation mode, presenting this to teacher-colleague? He drags G – again describing the diagram, it's there so you can change the size of the circle and see the relationship is always true. He states that he doesn't understand why the chord is there, what benefit it provides and that maybe he is missing something. He's given similar diagrams to pupils before (w/o the chord and G): he thinks pupils convince themselves that the angle at the centre is double that at the circumference – he drags B in a jerky, uncoordinated way, as if mimicking a pupil. He drags C upwards, towards B, past where CD is a diameter, so the internal shape is now a convex quadrilateral, angle at the centre is reflex. Robert seems to be dragging for himself now, he speaks more softly, almost to himself "what happens if it	Tr/TD, demonstrating dependency Fo/OSK, stating of angle at the centre theorem Tr/TD, demonstrating rounding errors; CUE, choosing numerical example Tr/TD, changing the size of circle; Co/MCC, seeing the relationship holds Fo/OSK, extraneous chord? Tr/CUR, assessing pedagogic benefit of chord Tr/CUR, choosing diagram with or w/o chord and G Fo/OSK, exploring for oneself Fo/OSK, recognising a

comes this way?" - he notes it still works (100, 200). He drags C further past B "and if that comes over there?" He pauses, saying it doesn't work anymore. Of course it does work, but GGb is measuring the 'wrong' angles. Robert doesn't stop to consider why it doesn't work, but continues, coming out of his brief reverie, saying he thinks diagrams like this are almost "too easy, too obvious", "like Catchphrase" – whilst he's saying this he's slipped back into dragging it like a pupil, dragging B in a jerky way, then D, then B and D again. (8.23) I ask him about how he'd demonstrate it to pupils. He asks about whether he's demonstrating the theorem to them or demonstrating how to use the diagram, so that they can explore it for themselves on a computer. I slightly obfuscate his question. Anyway, he starts to explain as if he were demonstrating the theorem. He'd have it set up with nice numbers (B=60, A=120) so that they could easily spot the relationship, near symmetrical, chord near horizontal, B near vertically above A. Before dragging B, he'd hover over the point and ask the pupils to predict what happens when he drags it to one side. Robert says he would pose the question so they didn't have the option of saying stay the same (does it get bigger or smaller?) in order to provide a bit of conflict for when he does drag B and the angle remains constant. He would then drag B "all the way round" the major arc CD, to see it never changes – he never gets very close to C or D. Robert then says, more softly, I can't remember what it does if I bring it through here, dragging B onto the minor arc. He says if he knew it was going to do that, he deliberately avoid it, in the hope that one of his pupils would ask. As he says this, Robert is dragging B in a peculiar way, sort of bouncing B on C and D, as if there is a sort of force field at C and D repelling his efforts to go through. When he does go "past" D, he rushes B past and onto the minor arc. When he's made diagrams like this before, he's forced B to remain on the major arc but now he wouldn't do this, he would want them to think about what's going to happen. Due to his initial chosen set up, both angles in D1 read 120 at the moment when B is on the minor arc. He wants the pupils to predict that the angle at B won't remain at 60 anymore. He adeptly adjusts some angle labels to his liking, without comment, without apparent thought, like a routine, reflex action. Robert recaps, he'd first establish that the angle at B doesn't vary in the major "segment" and then he'd start thinking about points C and D. He'd ask the pupils to make predictions about what happens to the angles when you drag D upwards? What about when angle at the centre =150 or 140, then what's B? He explains he tends to "ask before doing". He drops B too close to G and sorts out the ensuing technical difficulty with ease, returning B to a nice angle B=65, A=130.	case 'works' Cy/TI, resolving unexpected cases Fo/OSK, concluding a case doesn't 'work' Fo/AtT, not adhering to initial setup; Tr/UIM, modifying initial setup Tr/CUE, choosing numerical examples Tr/CUE, choosing geometric configuration Co/MCC, highlighting relationship between angles Tr/CUE, showing a 'family' of cases → Co/MCC Tr/CUE, avoiding extremes Cy/TI, recollecting cases as they arise Co/AC, anticipating a 'complex' case Co/MCC, drawing attention to variance Tr/CUE, limiting/allowing cases Tr/CUE, choosing special numerical example; UIM, changing initial set-up Co/DS, sequencing configurations Co/MCC, highlighting relationship between angles Tr/CUE, choosing numerical examples Co/MCC Tr/CUE, choosing numerical examples

(11.16) I ask about special cases. Robert immediately drags C so that CD is a diameter, saying he would show them "Thales theorem or whatever it is", the angle in a semi-circle as a special case of the angle at the centre. Then he wonders what else could we do? He says the case where B and C meet, become the same point, dragging B onto C as he does so. He then struggles, the angle at B slowly flips between 80 and 100 (A=160), asking "what does that mean?" to himself, pauses and then "I don't know what that means..." He doesn't resolve this difficulty, and moves on, recapping that he'd bring up the angle in a semi-circle and the case where B moves on to the minor arc.	Tr/CUE, choosing geometrical configuration Fo/UT, naming theorem Cy/TI, resolving unexpected case Fo/OSK, concluding case remains unresolved Tr/CUE, choosing geometric configurations

Appendix G. Excerpt from Robert's GeoGebra interview transcript

Rob-GGb-int, 13.6.2012, mins 5.05-12.10

I = interviewer, R = Robert.

I So what's going on in, what's going on with this diagram?

R Okay, so we've got, so the three red points are presumably all fixed to the circumference of the circle and I can modify all of them, so they're all free to move. Um, and what they do, so two of the points define a chord. The third point are connected to those two points and it … more about that chord, there's a subtended angle which is marked and measured and at the centre there's the same angle which in the way that it's currently looking is double, but not always if you choose numbers that round awkwardly. Um, and G means I can change the size of the circle and see that this relationship is always true. Um, the thing I thought when I was looking at this first time around was if the intention of this is to for pupils to see that the angle at the centre is double that at the circumference, then I wasn't necessarily sure about how actually drawing in the chord helped them with this. So that was one thought and that was on all three diagrams, so maybe there's a way of explaining this that I've not thought through. So this I mean I think each of these diagrams has merit and I think in different circumstances I think if … in the past I've given pupils this exact diagram minus the chord, although I think I've not let them change the size of the circle I think in the versions I've given them. And they kind of sit here and play with this for a bit and I think they convince themselves of the fact that this you know that no matter what I do with the exception of the rounding issue the angle at the centre is always double the angle at the circumference. What happens if it comes this way? That still works. And if that comes over there it doesn't work anymore. Um, yeah when I've given this to pupils I've never been convinced they've you know it almost seems too easy and too obvious is one thing.

I Mmm, mmm.

R And often like the activities I've seen and downloaded with circle theorems do just seem to be kind of you know like Catchphrase like say what you see you know.

I Okay, um, so if you were going to demonstrate …

R Yeah.

I … this to pupils, can you show me how you might drag the diagram?

R Um, so this is me demonstrating them to, demonstrate this circle theorem or me demonstrating them what to do so that when they get onto the computer themselves they can …? Are they going onto a computer themselves with this diagram or is this a I'm teaching them about circle theorems in class and this is me discussing?

I	I guess I'm going to ask you about what you would do in different circumstances.

R	Yeah.

I	So I think for this if you were demonstrating this diagram to them.

R	Okay.

I	Um, how would you drag it?

R	I'd probably I'd have it so when it came up I'd probably have it set up with I guess fairly nice numbers that they should be able to spot quite easily and I'd probably ask them what the relationship is. And then before dragging this point I'd probably you know I'd probably have it set up so that maybe it looks a bit well this is a kind of this is a nice symmetrical, that's horizontal, they almost look vertical you know. And so I'd probably ask them well what happens if I move this over here? Is it going to get bigger? Is it going to get smaller?

I	Can you show me?

R	So I probably would be if this was an interactive whiteboard I'd be hovering over this and not actually touching it and saying I'm going to drag this this way. What's going to happen?

I	Mmm, mmm.

R	And I'd probably try to lead them into, I probably wouldn't give them the option of it staying the same. I'd probably ask them is it going to get bigger or going to get smaller?

I	Mmm, mmm.

R	To I guess when they see that it does stay the same to provide a bit of conflict there. And then I'd drag it and we'd drag it all the way around here and show that it never changes. I can't remember what happens if I bring it over here. So I think if, I think if I knew it was going to do that I think I'd possibly … I think I'd probably deliberately avoid doing it and hoping that one of them would say well what happens if you go past here?

I	Mmm, mmm.

R	Because on diagrams I've had in the past I've forced it to just lie on the major arc, but I think now I wouldn't force that restriction, but I'd probably want them to think about what's going to happen. You know because I think they should hopefully be able to see that when it happens there's no way that that can be 60 degrees anymore. Um, and then at this point

I'd then … so I think the thing I'd vary first is I'd just establish that it doesn't matter where this point is in on the circumference of in that segment. And then I'd start thinking about these points and I'd probably again I'd probably pause and ask the question if I move this point round up here, what's going to happen to this angle and what do you think will happen to this angle? And hopefully, so if I change this angle here to be 150 degrees or let's say 140 degrees, what's going to happen to this angle here?

I So it's key you're asking them to predict first before …

R That's what I tend, that's what I tend to do now yeah before, so if this is as a whole class I tend to ask before doing.

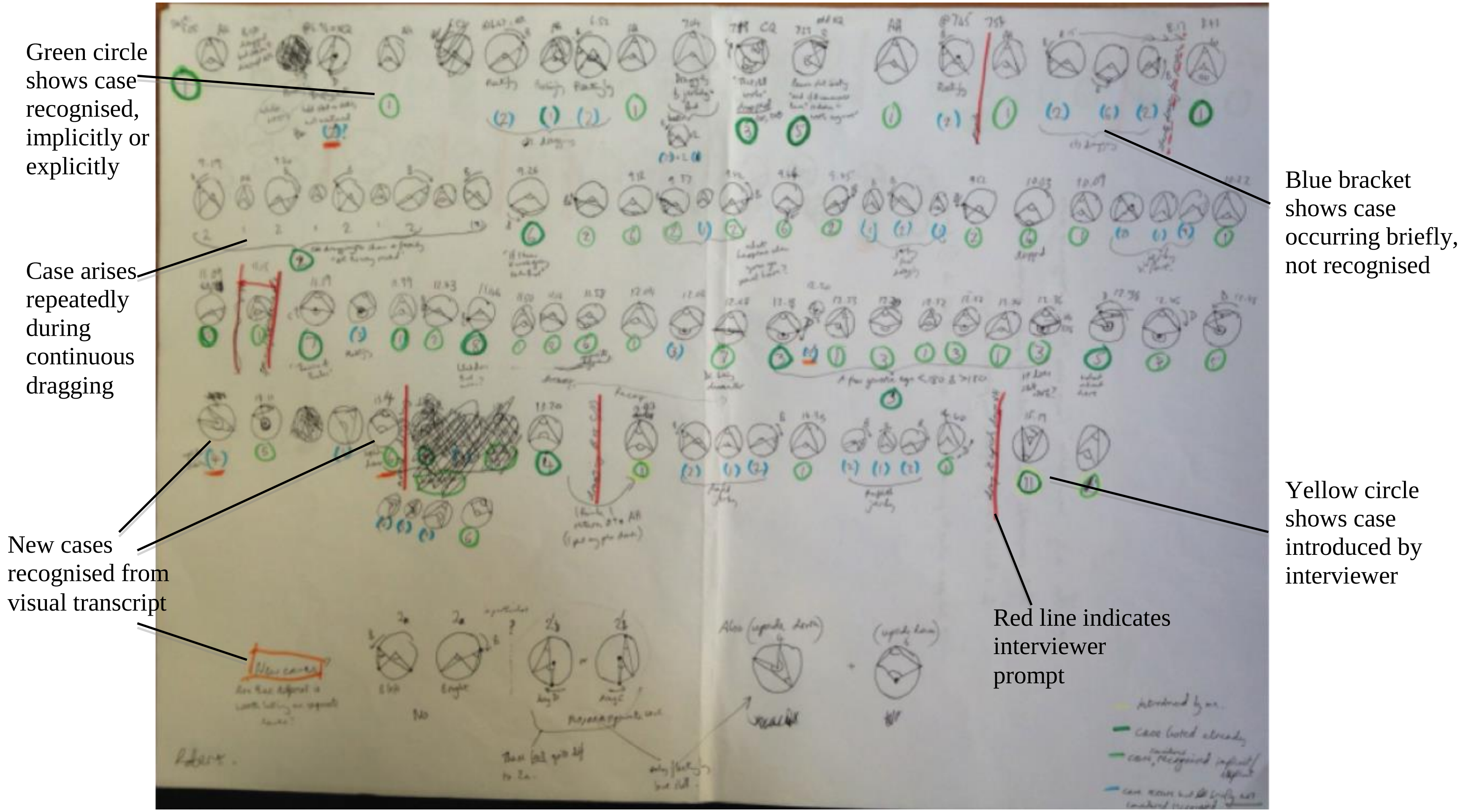
I Mmm, mmm, okay.

R I think.

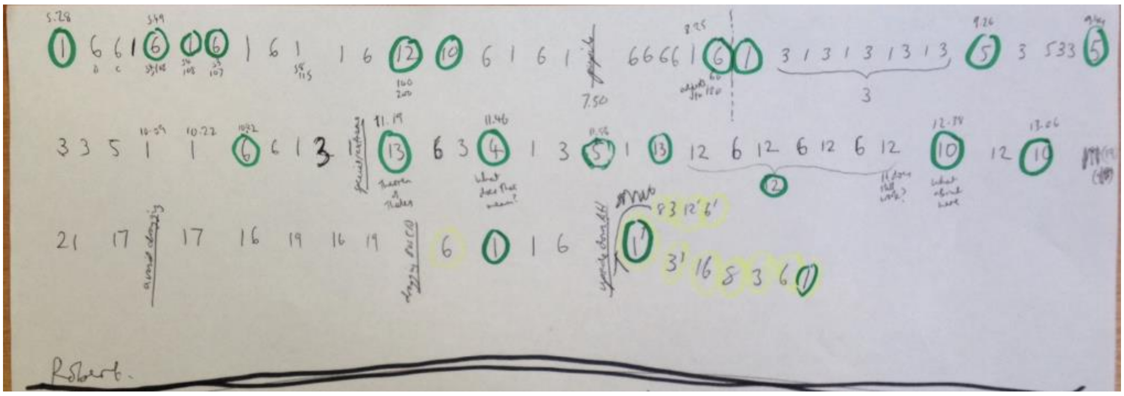
I So, um, so you kind of mentioned about dragging it round and then avoiding or trying to trigger something, so that's kind of like a special case I guess. Um, are there any other special cases or extreme cases that you would show to your pupils?

R Um, I'd probably want to show this. I'd probably want to show you know linking into ... what's it called theorem of Thales? ... the angle in a semi-circle is just really a special case of the angle at the circumference being half the angle at the centre. Um, I probably … what else could we do? So that's one. Um, I would if they didn't come up with it probably think about the case when this eventually gets to be you know B and C. This would be effectively the same point. I'm trying to think what that means. That would mean that they're … I don't know what that means. Yeah, I think I mean those are the two main ones I think in the opposite segment and also the D, C being a diameter.

Appendix H. Visual transcript



After the list of pedagogic cases was finalised and renumbered, see Figure 6.9, the video of Robert's GeoGebra was reviewed to produce the case list below. A similar process took place for the other three case study teachers. This time it did not seem necessary to sketch circle theorem configurations to form a visual transcript, since the pedagogic case list was assumed to be complete.



Here, the green circles indicate cases that were explicitly recognised; any other cases listed are implicitly recognised. The yellow circles again indicate cases arising due to the interviewer's dragging rather than the case study teacher's. The vertical dividing lines indicate the interviewer's verbal prompts. The bracketed cases indicate the case arises repeatedly during continuous dragging to show a 'family' of cases. This list was then typed and formatted according to the coding system in Table 6.1 and Figure 6.8 to produce a final case list.

Robert

1 **6** **12** **10** | ***3*** **5** | 13 4 *21* *17* | *16* *19* || *8* *12'* *6'* 1' *3'*

Appendix I. Triangulation across lesson observation and post-observation interview

The following excerpts have been chosen to indicate the triangulation process across lesson observation and post-observation interview, comparing teachers' espoused theories with their theories-in-action. The excerpts relate to a contingent moment in Edward's IWB lesson, where he has an insight with regard to self-inverse functions – detailed in Chapter 7. The post-observation interview clarifies the observation notes, showing that the moment was unplanned and represented a deviation from the planned lesson agenda. An excerpt from more detailed notes written shortly after the lesson observation and interview are also included.

<u>Excerpt from observation notes made in Edward's IWB lesson, Wednesday 20.6.2012</u>

Observation of year 12 'single maths' group, period 4 11.15-11.55 so 40 min lesson. The group was 'mixed-ability' – Edward described it as the most mixed ability maths class in the school, bar year 7 groups taught in form groups. To put this in some perspective, in 2012, no pupil in the school got below grade B in GCSE maths. The students were post-AS-level exams and Edward said he had increased the pace of lessons. Lesson content: graphing inverse functions, knowing it's a reflection in the line y=x, need to restrict the domain of some functions to ensure inverse exists.

ICT: Data projector (no IWB) onto a rotating whiteboard. Edward had manually installed GeoGebra onto the teacher computer in the classroom. The room was the second room change (one due to exams, one due to wanting to use ICT). WB is at the front, pupils sat in rows in pairs or threes in a 'traditional' classroom layout. GeoGebra file with function + point on function + trace of reflection of point on function in the line y=x

Incredibly rapid fire Q+A I'd guess sometimes less than 1 second to answer, even with ones they're meant to be working on in their books Goes thru' pupil: how can it be an inverse of itself? Edward: good question, gives example y=4-x and sketches graph [Did y=4-x come off the top of E's head or was it planned?] Big idea: self-inverse functions eg y=4-x have reflective symmetry in y=x pupil: oh cos perpendicular Edward: excellent point I think we should summarise Recaps some of the 'big ideas' he referred to including 'self-inverse function' Ran out of time to address	KQ code, Contingency: teacher insight, use of opportunity

Excerpt from post-observation interview on Edward's IWB lesson, Ed-IWB-int, 20.6.2012, mins 21.24 - 23.38

I = interviewer, E = Edward

I Okay right. And I was going to say you also booked this ... the room specially. Um ..

E Oh yeah because … uh … so every room has a computer at the front, but there was a room change today because the one we should have been in was being used for exams. And uh ... the room we were going to be in is a Classics room, which the set up is ... they never use computers in Classics ... so the setup is like ergonomically insane, so … the screen is like facing the wall and you have to stand on your head to use the computer, it's like it's not useful ... using that room ... notionally having a computer in the room.

I Yeah yeah okay. I had a question now that … as you said there was a neat thing when at the end of the lesson one of the boys … he got one of the questions wrong, and he sort of went 'Yeah I've got it wrong' and then the answer turned out to be $Y = 4 – X$

E He actually got it right, but he thought it was wrong.

I Oh okay okay. And uh … yeah, and he said 'Wow how can it be the same thing?'

E How can the inverse of $Y = 4 – X$ be the same thing be the same thing as $Y = 4 – X$

I Yeah yeah. And I was going to ask you, did that example come off the top of your head, but actually I misunderstood, it was actually the example in the book.

E It was a question from the book, but I hadn't looked at it before and I was stuck for a bit, it's like is that right or wrong ... oh that means this and I realised it was self-inverse and reflective.

I Yeah, okay.

E Which could have been a really good learning objective I just hadn't thought of it. So that's another example of the book coming up with points that I wouldn't have thought of. So it's not all bad, the questions in the book, they are a bit random sometimes. In fact that wasn't, that was quite a good question to ask actually.

I Good. Um … there was something in the lesson that I had a bit of a stumble with … or again I just wanted to get my head round it with you. One of the girls asked something like how many inverse functions are there for the $X^3 – 2X +1$, and it was trying to understand the conversation that went round that, cos you pointed to the $f(X) = X^3 – 2X + 1$, and you said something like it's not a function, it's a mapping.

Excerpt from notes typed up on Wednesday 20.6.2012, after Edward's IWB lesson observation and post-observation interview

The pupils then begin work on the questions from the textbook and I guess they have about 5 mins to do this and then he starts going through the answers to the first question. Edward was going to introduce a final example when the vocal boy at the side has a problem with one of the answers – he thinks he has it wrong, how can the inverse of y=4-x be y=4-x? He asks how can it be an inverse of itself?

Edward answers it's a good question and draws the line on the board, noting it's perpendicular to y=x and thus has reflective symmetry in y=x. This was clearly an unplanned moment, which I thought Edward dealt quite nicely with. He includes this in his summary of the big ideas of the lesson at what is now the end of the lesson.

Asking him in interview, the example he was going to look at was y=x^2 – 3 which was one of the questions he had given them – a boy at the back that I talked to and who didn't contribute during the lesson came a bit unstuck on this when I probed him. Edward had also come across a pupil having difficulty with this one and so thought he'd address it at the board, but he didn't have time in the end.

Appendix J. Transcription protocol

When it is clear from the surrounding text in a chapter who the speaker is, then an initial is not provided for identifying the speaker. When clarification seems necessary, an initial is provided, separated from speech by a colon and indentation.

Speaker initials: I = interviewer, M = Michael, E = Edward, A = Anne, R = Robert

Pauses are indicated by three full stops ... in the text of the speech.

Square brackets [] are used to indicate where extra words have been added by the author for clarification or to provide extra information for context.

Speech marks ‘’ within a quote are used when the speaker suggests what they *would* say, for example, in a classroom context.

<u>Underline</u> indicates the speaker’s heavy emphasis on the words so indicated.

“Er”, “um” or “uh” are used as they sounded to indicate utterances where the speaker is hesitating. “Mmm” indicates an utterance, usually a non-question prompt from the interviewer to the respondent to continue talking.

Punctuation has been used in a conventional sense: question marks have been used when a raised note suggests a question is being asked and exclamation marks indicate surprise or excitement.

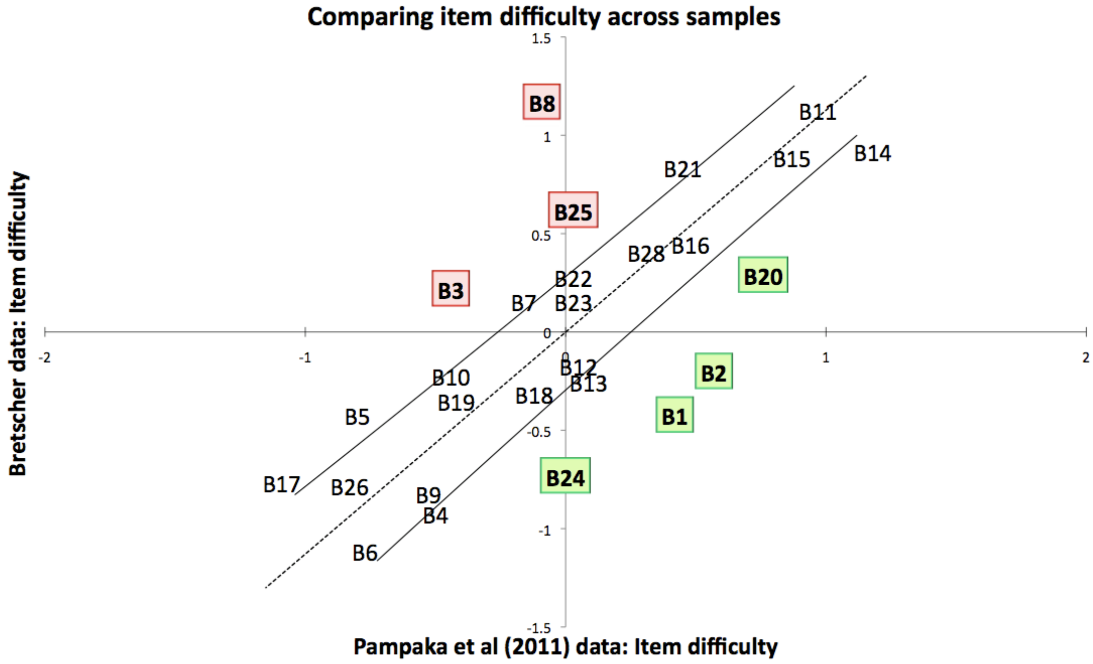
Spoken mathematics is recorded as mathematical notation. For example, where numbers are spoken to convey an angle, “140 degrees” is recorded. Similarly, when formulae or functions are spoken this is recorded in mathematical notation e.g. “$f(x) = 4 - x$”.

When the recording was not good enough for the speech to be heard, this is indicated by (inaudible).

Indicative examples used in Chapter 5:

> E: [later in the interview] I’d definitely mention it because sometimes it doesn’t seem to work does it? ...it was one degree out... but I <u>really</u> see it as a hindrance to learning what’s going on. I’d just, I’d have to keep saying ‘Look, within rounding error this result is...’, sort of, it’s much less convincing... [Ed-GGb-int, 20.6.2012]
>
> M: Um, I guess it’s [D3 is] useful in that it shows that the rules only apply on the circumference or certainly that particular rule of the double angle at the centre, circumference etc.
>
> I: Mmm, mmm.
>
> M: Um, so I guess it’s useful to show that that [D3] doesn’t work, whereas that [D1] does work. [Mic-GGb-int, 31.5.2012]

Appendix K. Scatter graph comparing item measures



Appendix L. Ethical approval

Nicola Bretscher
Department of Educational and Professional Studies
28th September 2011

Dear Nicola,

REP(EM)/10/11-74 – 'Using ICT to Investigate Mathematical Knowledge in Teaching.'

I am pleased to inform you that the above application has been reviewed by the E&M Research Ethics Panel that FULL APPROVAL is now granted.

Please ensure that you follow all relevant guidance as laid out in the King's College London *Guidelines on Good Practice in Academic Research* (http://www.kcl.ac.uk/college/policyzone/attachments/good_practice_May_08_FINAL.pdf).

For your information ethical approval is granted until 27th September 2013. If you need approval beyond this point you will need to apply for an extension to approval at least two weeks prior to this explaining why the extension is needed, (please note however that a full re-application will not be necessary unless the protocol has changed). You should also note that if your approval is for one year, you will not be sent a reminder when it is due to lapse.

If you do not start the project within three months of this letter please contact the Research Ethics Office. Should you need to modify the project or request an extension to approval you will need approval for this and should follow the guidance relating to modifying approved applications: http://www.kcl.ac.uk/research/ethics/applicants/modifications.html

Any unforeseen ethical problems arising during the course of the project should be reported to the approving committee/panel. In the event of an untoward event or an adverse reaction a full report must be made to the Chairman of the approving committee/review panel within one week of the incident.

Please would you also note that we may, for the purposes of audit, contact you from time to time to ascertain the status of your research.

If you have any query about any aspect of this ethical approval, please contact your panel/committee administrator in the first instance (http://www.kcl.ac.uk/research/ethics/contacts.html). We wish you every success with this work.

Yours sincerely

Daniel Butcher
Research Ethics Administrator

Amendments

Changes to the questionnaire and accompanying information sheet

- The information sheet has been altered to say completion implies consent. Similarly a sentence has been added at the start of the questionnaire indicating completion implies consent.
- A section has been added to the end of the questionnaire so that participants wishing to be involved in any further part of my case study can tick a relevant box and leave their email address or equivalent.
- Appropriate changes have been made to the section on Informed consent on the application form to reflect the changes above.

Regarding anonymity issues in returning the questionnaire to the HoD.
Following the same procedure as for my pilot for this survey, a blank envelope is attached to each questionnaire. The participants are instructed on the information sheet to place completed questionnaires in the envelope and seal it, before returning it to the HoD. Thus at no point should the HoD have access to participants' data. This should ensure participants anonymity and the confidentiality of their data.

I confirm that I will not be video or audio recording classroom observations.

Informing students of observations and opting out if they do not wish to take part
Although not directly participant, some students will nevertheless be incidental to the study. At the start of a lesson prior to the lesson under observation, the researcher will outline the study in brief as one of studying their teacher's use of ICT to teach mathematics and invite them to raise questions with the researcher if they have any concern. In particular, they will be reassured that they are not the object of the observation and no student will be identifiable from any written account of the study. It will be possible to proceed with the study while avoiding a given small number of students if that is what they wish, without compromising the data.

Research Ethics – Risk Checklist

- Complete the checklist ticking yes to any of the sections relevant to your study.
- Submit the checklist along with your application to the committee, ensuring each copy of the application has a checklist attached on top.

Name:	Nicola Bretscher
Review Committee:	Education and Management Research Ethics Panel
Title of Study:	Using ICT to investigate Mathematical Knowledge in Teaching

		Yes	No
A	Does the study involve **participants who are particularly vulnerable** or unable to give informed consent or in a dependent position (e.g. children, your own students, over-researched groups, people with learning difficulties, people with mental health problems, young offenders, people in care facilities, including prisons)? **If you have ticked yes to this section,** will **financial incentives** (other than expenses) be offered to participants? **YES** ☐ **NO** ☐ **If yes, please state how much.**	☐	☒
B	Will participants be asked to take part in the study without their consent or knowledge at the time or will **deception** of any sort be involved (e.g. covert observation of people in non-public places)?	☐	☒
C	Is there a risk that the highly sensitive nature of the research topic might lead to **disclosures** from the participant concerning their own involvement in illegal activities or other activities that represent a threat to themselves or others (e.g. sexual activity, drug use, or professional misconduct)?	☐	☒
D	Could the study induce **psychological stress or anxiety**, or produce humiliation or cause harm or negative consequences beyond the risks encountered in normal life?	☐	☒
E	Does the study involve **physically intrusive** procedures? **If yes, continue below:**	☐	☒
i	Does the study involve **only moderately** intrusive procedures *(taking less than 40ml blood, collecting bodily waste, cheek swabs)*?	☐	☐
ii	Are substances to be administered (such as **food substances**) which are not classified as 'medicinal products' by the MHRA? (see 15c of the guidelines for more details)	☐	☐
iii	Are substances which are classified as **'medicinal products'** by the MHRA to be administered? (see 15c of the guidelines for more details)	☐	☐
iv	Does the study involve **imaging techniques** such as MRI scans, x-rays or ultrasound?	☐	☐
v	Does the study involve **DNA or RNA** analysis of any kind? (see Appendix D)?	☐	☐
vi	Are invasive, intrusive or potentially harmful procedures **not already covered** by items i, ii, iii, iv, & v to be used in this study?	☐	☐

KING'S College LONDON

For office use only:
REC Protocol No
Date rec'd:

APPLICATION FOR ETHICAL APPROVAL

Please tick the Committee you are applying to:

Sub-Committees (RESC)

PNM RESC ☐
(Psychiatry, Nursing & Midwifery)

SSHL RESC ☐
(Social Sciences, Humanities & Law High Risk)

BDM RESC (Health) ☐
(Biomedical & Health Sciences, Dentistry, Medicine and Natural & Mathematical Sciences)

Research Ethics Panels (REP)

For SSPP, Humanities and Law (non-high risk only)

E&M REP ☒
(Education & Management)

GGS REP ☐
(Geography, Gerontology, SCWRU)

Humanities REP ☐

War Studies Group REP ☐

Law REP ☐
(Law & Department of Political Economy)

Notes for all applicants

- Please read the guidelines before filling out the application form and refer to the specific guidelines about each section when filling in the form. (http://www.kcl.ac.uk/research/ethics/applicants/)
- Refer to the Guidelines for the submission deadlines for your Committee and the number of copies to submit (including electronic versions if applicable).
- All applications should be submitted **by 5pm on the deadline day.**
- All Sub-committee applications should be submitted to the Research Ethics Office, 5.11 Franklin Wilkins Building, (Waterloo Bridge Wing), Waterloo Campus, King's College London, Stamford Street, London SE1 9NH.
- All Research Ethics Panel applications should be submitted to SSPP Ethics Administrator, K0.58 Ground Floor Strand Building, King's College London, The Strand, London WC2R 2LS.

SECTION A – TO BE COMPLETED BY ALL APPLICANTS

1. APPLICANT DETAILS
1.1 RESEARCHER
Researcher's Name: Nicola Bretscher **Researcher's Department & School:** Department of Education and Professional Studies, School of Social Science and Public Policy **Status:** ☐ Undergraduate ☐ Taught Postgraduate ☒ MPhil / PhD/ Specialist Doctorate ☐ Staff Research **If Student:** **Name of course/qualification:** MPhil/PhD in Education **If Staff:** **Researcher's Post:** N/A
1.2 CONTACT DETAILS
Email: nicola.bretscher@kcl.ac.uk **Telephone number:** 07957 242433 **Address:** 13 Derby Hill Crescent, Forest Hill, London SE23 3YL
1.3 SUPERVISOR - COMPLETE FOR ALL STUDENT PROJECTS (Including PhD)
Name of Supervisor: Jeremy Hodgen **Supervisor's Post:** Senior Lecturer in Education **Supervisor's Department (if different to student):** DEPS **Supervisor's email address:** jeremy.hodgen@kcl.ac.uk
1.4 OTHER INVESTIGATORS, COLLABORATORS, ORGANISATIONS
List any other investigators/collaborators involved with the study, and ensure that their role (e.g. collaborator, gatekeeper) and responsibilities within the project are explained. You should include any draft/preliminary approach letters to gatekeeper organisations and confirm that you will have permission letters available for inspection if requested for audit purposes. **NB:** For other investigators/collaborators clarify if their employer is not King's College London. N/A

2. PROJECT DETAILS	
2.1 Project Title	Using ICT to investigate Mathematical Knowledge in Teaching
2.2 Projected Start Date of Project This should be when you intend to start work with participants	September 2011
2.3 Expected Completion Date of Project	April 2013
2.4 Sponsoring Organisation Your sponsor will be assumed to be King's College London unless stated otherwise. **NB: Do not put 'N/A'**	
2.5 Funder (e.g. self-funded, King's College London, ESRC, AHRB, EU)	ESRC

2.6 DOES THE STUDY INVOLVE HUMAN PARTICIPANTS OR FOR OTHER REASONS REQUIRE ETHICAL APPROVAL?
NB: It may be the case that research does not involve human participants yet raises other ethical issues with potential social or environmental implications. In this case you should still apply. Please consult with the Research Ethics Office (rec@kcl.ac.uk) if in doubt. ☒ **Yes** ☐ **No**
2.7 OTHER INFORMATION RELATING TO RISK
Will the study place the researcher at any risk greater than that encountered in his/her daily life? (e.g. interviewing alone or in dangerous circumstances, or data collection outside the UK). **Yes** ☐ **No** ☒ **If applicable:** **Does the study involve the using a Medical Device outside of the CE mark approved method of use?** (see guidelines) If you are using a medical device 'off label' (outside of the approved method of use) then a risk assessment needs to be completed. For further information on medical devices see the Medicines and Healthcare Products Regulatory Agency webpages: http://www.mhra.gov.uk/Publications/Regulatoryguidance/Devices/index.htm and http://www.mhra.gov.uk/Publications/Regulatoryguidance/Devices/GuidanceontheECMedicalDevicesDirectives/index.htm. **Yes** ☐ **No** ☒ **If you have ticked yes to either of the above:** ☐ **Yes**, and I have completed a risk assessment which has been co-signed by the Head of Department/ I have discussed the risks involved with my supervisor or Head of Department and agreed a strategy for minimising these risks.
2.8 OTHER PERMISSIONS, ETHICAL APPROVALS & CRIMINAL RECORDS BUREAU CLEARANCE REQUIRED
ANOTHER REVIEWING BODY/PERMISSIONS - Are any other approvals by another reviewing body (<u>including other ethics committees, gatekeepers and peer review</u>) required? If yes, give details and say when these will be obtained. If they have already been obtained you should provide a copy of the approval with the application otherwise you will need to supply it when ready. **YES** ☐ **NO** ☒ **CRIMINAL RECORDS BUREAU** - Is Criminal Records Bureau clearance necessary? If so, please confirm that clearance will be sought before commencement of the project. **YES** ☐ **NA** ☒
2.9 HUMAN TRIALS QUESTIONNAIRE
Does a **human trials questionnaire** need to be submitted? **YES** ☐ **NO** ☒ (http://kcl.ac.uk/about/structure/admin/finance/staff/insurance/trials.html) If yes, confirm that the Human Trials Questionnaire will be submitted prior to the start of the study. **YES** ☐

3. AIMS, OBJECTIVES & NATURE OF STUDY

This study aims to investigate mathematical knowledge in teaching using Information and Communication Technologies (ICT). In a major review of research on mathematical knowledge in teaching (Rowland and Ruthven, 2011), not one of the papers discusses teacher knowledge directly in relation to ICT. Using ICT as a means for conceptualising, exploring or developing teachers' mathematical knowledge is not discussed. Neither is the knowledge needed to teach mathematics effectively using ICT explored. This is surprising given the emphasis on using ICT in the UK National Curriculum for mathematics and the inclusion of ICT in mathematics curricula more globally, coupled with widespread recognition of the complexity of integrating ICT into teachers' practice (eg Ruthven, 2007). This study aims to address these issues by developing well-researched tools (Williams, 2011) involving ICT to contribute towards answering the following research questions:

- What is the nature of mathematical knowledge needed for teaching mathematics using ICT?
- How is this knowledge actualised in teaching mathematics using ICT?

Hodgen (2011) argues the case for conceptualising the nature of mathematical knowledge in teaching as situated, social and distributed. However, Ruthven (2011, p.87) notes that the evidential basis for adopting such a perspective remains slender and fragmentary. Williams investigated teacher knowledge in relation to graphs, comparing a test instrument involving paper and pencil tasks to knowledge elicited through interviews. He concluded that teacher knowledge was task and tool dependent and thus teacher knowledge is 'distributed'. By using ICT as a tool for investigating teachers' mathematical knowledge in varying personal and social contexts, this study hopes to contribute to an understanding of mathematical knowledge in teaching as situated, social and distributed.

4. STUDY DESIGN/METHODOLOGY, DATA COLLECTION & ANALYSIS

Provide a brief outline of the step-by-step procedure of your proposed study, in no more than 1 page where possible. (An example of a flow chart that could be used is in the Guidelines.)

The research project will employ mixed methods in terms of a large-scale survey of teachers' ICT use alongside in-depth case studies. Participants in the survey will fill out a questionnaire, which should take no longer than 30 minutes to complete. Teachers are free to complete the questionnaire at a time and place of their choosing. Four case studies of individual teachers will be selected using information gained from the questionnaires. Case study teachers will be expected to take part in three interviews and two classroom observations. Interviews should last no more than one hour. Interviews and classroom observations will take place at the participant's school, at a time of their choosing. Please see sample questionnaire and interview guides for further details.

The survey will provide a broad overview of ordinary mathematics teachers' ICT use. In addition, the data will be used to investigate patterns of use according to departmental and individual factors. The survey instrument will seek background information on personal and contextual factors, which will be used to inform case study selection. In particular, Pampaka et al's (2010) items will be used, with possible adaptations, to measure teachers' pedagogic orientations on a connectionist-transmissionist scale. The questionnaire will also seek information on contextual factors: the degree of integration of ICT into schemes of work and the priority given to ICT within the department. These measures give a sense of the level of support for using ICT within the department. The survey will be used to develop an improved conceptualisation of connectionist and transmission orientations in relation to ICT use and thus the mathematical knowledge in teaching associated with these orientations. Based on the survey data, four case studies will be selected according to the following criteria:

Connectionist teacher in dept supportive of ICT use	Transmission teacher in dept supportive of ICT use
Connectionist teacher in dept unsupportive of ICT use	Transmission teacher in dept unsupportive of ICT use

Drawing on previous research (Askew et al, 1997; Bibby, 2002), data collection in case studies will involve classroom observation, structured and semi-structured interviews. Viewing teachers' knowledge as situated, collecting data in varied situations will provide different windows on teachers' knowledge (Adler, 2001). Classroom observation gives access to the knowledge teachers' use as they respond to contingencies arising in the immediacy of classroom practice. ICT use in alternative classroom contexts (whole-class instruction versus student computer work) may provoke different contingencies, giving access to different facets of teachers' mathematical knowledge. Interview data provide another important window on mathematical knowledge in teaching. For example, interviews may highlight teachers' knowledge of mathematics for professional use outside of the classroom, and their knowledge of pupils' mathematical learning. Screencasts will be collected during interviews and observations to provide visual record of teachers computer work. The types of data collection within case studies are listed below:

- Classroom observation, whole-class instruction
- Classroom observation, direct student access to computers
- Personal construct interview– knowledge of pupils' mathematical learning in relation to ICT
- Concept mapping interview – mathematical knowledge in relation to ICT
- Background interview – to provide more detail on survey items

Questionnaire data will be analysed using descriptive and inferential statistics (eg t-tests and Chi-squared tests) where appropriate. Open-ended responses will be analysed manually. Interviews will be transcribed and coded for qualitative analysis. Classroom observations will focus on the teacher, their teaching styles, resources and management and organisational strategies, to produce detailed accounts of the flow, content and context of the lesson (Askew et al, 1997). These accounts will be coded for qualitative analysis. Advice on data analysis has been sought primarily from my supervisor.

5. PARTICIPANTS TO BE STUDIED

5.1 PROJECTED NUMBER OF PARTICIPANTS

Number: Estimated 125 survey participants; 4 individual case study teachers

If applicable: How many will be male and female. N/A

Justification for the sample size: A sample of 125 mathematics teachers should provide sufficient and manageable data for statistical analysis, based on research using similar measures eg Pampaka et al (2010). 100 schools will be approached to take part in the survey. Assuming a response rate of 25% and an average of 5 teachers per school participating (based on pilot survey results), 125 participants should be achievable. Four case studies will be selected to provide contrasting situations, based on personal and contextual factors, in which to investigate teachers' mathematical knowledge in teaching using ICT.

If an upper age limit is needed you must provide a justification.

Upper Age Limit: N/A **Lower age limit:** N/A

5.2 SELECTION CRITERIA

Survey participants must be actively teaching mathematics in a UK secondary school. Schools approached to participate in the survey must be secondary schools offering mathematics across Key stages 3 and 4. Case study teachers must be actively teaching mathematics in a UK secondary school with some access to ICT resources and willing to be observed using ICT in their teaching. In addition, they should fulfil one set of case study criteria, for example, have a connectionist orientation in a mathematics department that is unsupportive of ICT use.

5.3 RECRUITMENT

Describe how participants will be (i) identified and (ii) approached.

The schools will be identified via existing contacts with King's College London, for example, their participation in the secondary mathematics PGCE course. The head of mathematics of each school will be approached by letter and email initially and their permission sought to distribute questionnaires to each member of staff in their department (see head of department letter). Potential case study teachers will be identified via their questionnaire data. They will be approached via letter and email (see teacher's letter).

5.4 LOCATION

State where the work will be carried out e.g. public place, in researcher's office, in private office at organisation.

Teachers are free to complete the questionnaire at a time and place of their choosing. Classroom observations and interviews will take place at the participant's school, at a time of their choosing.

6. ETHICAL CONSIDERATIONS

6.1 INFORMED CONSENT
Describe the process you will use to ensure your participants are freely giving fully informed consent to participate. This will **always** include the provision of an information sheet and will normally require a consent form unless it is a purely self-completion questionnaire based study or there is a justification for not doing so (this must be clearly stated). Templates for these are at the end of this document and they should be filled in and modified where necessary. A consent form will be attached with the letter seeking permission from the head of department. Since the questionnaire is purely self-completion and anonymous, completion will be taken to imply consent – participants are informed of this on the questionnaire and information sheet. There will be a separate information sheet and consent form for potential case study teachers, attached to the letter seeking their participation in the case study.
6.2 RIGHT OF WITHDRAWAL
(Participants should be able to withdraw from the research process at any time and also should be able to withdraw their data if it is identifiable as theirs and should be told when this will no longer be possible (e.g. once it has been included in the final report). Please describe the exact arrangements for withdrawal from participation and withdrawal of data depending on your study design). The head of department is informed in the letter seeking their permission that their school's participation is voluntary and that they may withdraw from the study at any time. Questionnaires will be linked to the participating school via a coded reference, enabling me to withdraw all the data from a participating school should the head of department request it. Participants can contact me via e-mail to state that they no longer wish to take part. Case study teachers are informed that their participation is voluntary and that they may withdraw from the study at any time in the letter seeking their participation and on the attached information sheet. Participants may request the withdrawal of their data up until the date of first publication/dissemination of research findings. This is envisaged to be 1/2/2012 for survey data and 1/6/2012 for case study data.
6.3 RISK CHECKLIST
Where you have ticked **'Yes' on the risk checklist**, provide details of relevant qualifications and experience **with reference to those sections**. This **must** include the researcher and/or supervisor as well as other collaborators (if applicable) involved in those sections marked as presenting risk. (Do not submit a c.v.) N/A **You must also specifically address the ethical issues raised from those sections here**. N/A **NB: If you ticked yes to any point in E i -vi of the checklist, you must also complete and submit Section B of the application form.**
6.4 OTHER ETHICAL ISSUES
Please consider whether there are other ethical issues you should be covering here. Further, if **applicable**, please also add the professional code of conduct you intend to follow in your research. http://www.kcl.ac.uk/research/ethics/training/codes.html The project adheres to the British Educational Research Association's Revised Ethical Guidelines for Educational Research (2004) http://www.bera.ac.uk/files/2008/09/ethica1.pdf, see also http://www.bera.ac.uk/ethics-and-educational-research-/ and the King's College London Guidelines on Good Practice in Academic Research http://www.kcl.ac.uk/college/policyzone/index.php?id=247
6.5 BENEFITS & RISKS
Please describe any expected benefits and risks to the research participant. **For example:** **Will participants receive a copy of the final report?** **What is the potential for adverse effects resulting from study participation, e.g.** • **participants suffering pain, discomfort, distress, inconvenience or changes to lifestyle.** • **sensitive, embarrassing or upsetting topics being discussed/raised.** Identify the potential for each of above and state how you will minimise risk and deal with any untoward incidents/adverse

reactions.

Participation in the either the survey or as a case study may make participants more aware of how they use ICT resources in their teaching. In this sense, participation in the research may provide an opportunity for reflection on professional practice which participants may find beneficial. At the end of the study participants will be able to receive information about the results of the survey, should they wish. Neither the survey nor case study research involves the collection of sensitive personal or institutional information. Participation in the research will not disadvantage participants in any way and there is no risk, discomfort or inconvenience (minor or major) involved.

6.6 CRIMINAL OR OTHER DISCLOSURES REQUIRING ACTION

Is it possible that criminal or other disclosures requiring action (e.g. evidence of professional misconduct) could be made during this study?

YES ☐ **NO** ☒

If yes, detail what procedures will be put in place to deal with these issues. The Information Sheet should make it clear under which circumstances action may be taken by the researcher.

N/A

7 FINANCIAL INCENTIVES, EXPENSES AND COMPENSATION

7.1 Will travelling expenses be given? If yes, this should be stated on the Information Sheet

YES ☐ **NO** ☒

7.2 Is any reward, apart from travelling expenses to be given to participants? If yes, please provide details and a justification for this. It is recommended that participants are informed of the compensation on the information sheet.

YES ☐ **NO** ☒

7.3 Is the study in collaboration with a pharmaceutical company or an equipment or medical device manufacturer? If yes, please give the name of the company and indicate what arrangements exist for compensating patients or healthy volunteers for adverse effects resulting from their participation in the study (in most cases, the Committee will only approve protocols if the pharmaceutical company involved confirms that it abides by APBI (The Association of the British Pharmaceutical Industry) guidelines. A copy of the indemnification form (Appendix C) should be submitted with the application.

YES ☐ **NO** ☒

7.4 No fault compensation scheme If your study is based in the UK you **must** offer the No-fault compensation scheme to participants unless there is a clear justification for not doing so (if this is the case this must be stated and you should bear in mind that the Sub-Committee reserves the right to make this a condition of approval).

YES, I am making the scheme available to participants ☒

NO, the study is based outside the UK and so the scheme is not applicable ☐

NO, the study is within the UK but the No-fault compensation scheme is not offered for the following reason:

8. DATA PROTECTION, CONFIDENTIALITY, AND DATA AND RECORDS MANAGEMENT

8.a. Confirm that all processing of personal information related to the study will be in full compliance with the Data Protection Act 1998 (DPA) *including the Data Protection Principles).*

If you are processing any personal information outside of the European Economic Area you **must** explain how compliance with the DPA will be ensured.

YES ☒ **NO** ☐

8b. What steps will be taken to ensure the confidentiality of personal information? Give details of anonymisation procedures and of physical and technical security measures. Please note to make data truly anonymous **all information** that could potentially identify a participant needs to be removed in addition to names. **NB: Personal data held on mobile devices must be encrypted** (http://www.kcl.ac.uk/college/policyzone/assets/files/information_policies/Encryption%20Policy%2020100513FINALv1%20(2).pdf).

Participants are given explicit assurance of confidentiality. With the use of proper anonymisation techniques, for example by giving each school a coded reference, I ensure that participants data will not be associated with any specific individual, department or institution. Individual teachers will only be identifiable via a numeric identifier, known only to me. Lists linking schools with their coded reference and individual teachers with their numeric identifiers will be stored securely and separately from the data. Pseudonyms will be created for case study teachers. Participants will not be identified in any publication.

8c. Who will have access to personal information relating to this study? Confirm that any necessary wider disclosures of personal information (for instance to colleagues beyond the study team, translators, transcribers auditors etc) have been properly explained to study participants.

My supervisors and I will have access to the data collected. Interview data will be transcribed by the researcher (me). Thus, for the purposes of this study, no disclosures of personal information beyond the study team are planned.

8d. Data and records management responsibilities during the study. The 'Principal Investigator' is the named researcher for staff projects and the supervisor for student projects.

I confirm that the Principal Investigator will take full responsibility for ensuring appropriate storage and security for all study information including research data, consent forms and administrative records and that, where appropriate, the necessary arrangements will be made in order to process copyright material lawfully.

YES ☒ **NO** ☐

8e. Data management responsibilities after the study.

State **how long** study information including research data, consent forms and administrative records will be retained, **what format(s)** the information will be kept in and **where** the data will be stored. For example, where within King's College London? (http://www.kcl.ac.uk/iss/igc/tools/researchers.html)

Completed questionnaires, consent forms, observation records and transcripts will be retained for seven years after the study within the Department of Education and Professional Studies. Data extracted from the questionnaires will be stored electronically and kept for the same period. Recorded observation and interview material will be deleted after transcription.

NB: Any personally identifiable data that is held on any mobile device should be encrypted. This includes data stored on USB keys, laptop/netbooks, desktop computers, smart phones, workgroup servers and relevant emails.

In addition, confirm whether the storage arrangements comply with the Data Protection Act 1998 and the College guidelines.

YES ☒ **NO** ☐

Will data be archived for use by other researchers?

NO ☒

YES (in anonymised form) ☐ If you intend to share anonymised data with other researchers, you must make this clear on the information sheet.

YES (in identifiable form following the guidance below) ☐

Will any personal information related to this study be retained and shared in unanonymised form? If you tick yes you must ensure that these arrangements are detailed in the Information Sheet and that participant consent will be in place.

YES ☐ **NO** ☒

9. AUTHORISING SIGNATURES

9.1 RESEARCHER/APPLICANT

I undertake to abide by accepted ethical principles and appropriate code(s) of practice in carrying out this study. The information supplied above is to the best of my knowledge accurate. I have read the Application Guidelines and clearly understand my obligations and the rights of participants, particularly in so far as to obtaining valid consent. I understand that I must not commence research with human participants until I

have received full approval from the ethics committee.

Signature ....Nicola Bretscher............... Date.........13/6/2011.......................

9.2 SUPERVISOR AUTHORISATION FOR STUDENT PROJECTS (including PhD)

I confirm that I have read this application and will be acting as the student researcher's supervisor for this project. The proposal is viable and the student has appropriate skills to undertake the research. The Information Sheet and recruitment procedures for obtaining informed consent are appropriate and the ethical issues arising from the project have been addressed in the application. I understand that research with human participants must not commence without full approval from the ethics committee.

<u>If applicable:</u>

The student has read an appropriate professional code of ethical practice ☒

The student has completed a risk assessment form ☒

Name of Supervisor: Jeremy Hodgen

Signature ...... Jeremy Hodgen **............ Date............21/6/2011...................**

9.3 MEDICAL SUPERVISION (if appropriate – see the Guidelines)

Name of Medical Supervisor: N/A

Medical Supervisor's MDU/MPS (or other insurance provider) number:

....

Signature of Medical Supervisor:

.... **Date.................................**

10. INFORMATION SHEET AND CONSENT FORM

Remember to submit your information sheets for participants and consent form (if necessary) with your application. Failure to do so will cause delays to your applications.

The information sheet for participants should be composed according to the guidelines. **The text in red should be deleted or modified as appropriate. If the language in the template is not suitable for your intended participant group it can be modified.** There is also a template consent form that can be used. Please refer to the guidelines for further information on how these documents should be used.

REFERENCES

Adler, J. (2001). *Teaching Mathematics in Multilingual Classrooms*. Dordrecht: Kluwer.

Askew, M., Rhodes, V., Brown, M., Wiliam, D., & Johnson, D. (1997). *Effective Teachers of Numeracy*. London: King's College.

Bibby, T. (2002). Shame: an emotional response to doing mathematics as an adult and a teacher. *British Educational Research Journal, 28*(5), 705-721.

Hodgen, J. (2011). Knowing and Identity: A Situated Theory of Mathematics Knowledge in Teaching. In T. Rowland & K. Ruthven (Eds.), *Mathematical Knowledge in Teaching* (pp. 27-42). London: Springer.

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KING'S College LONDON
University of London

[date]

Dear Head of Mathematics

Using ICT to investigate Mathematical Knowledge in Teaching

I am writing to invite your department to participate in a survey as part of the main study for my PhD research project at the Department of Education, King's College London. This project aims to investigate the mathematical knowledge that teachers draw upon in their teaching of mathematics using ICT. In this project ICT is also used as a tool to explore teachers' mathematical knowledge.

If you choose to participate in my survey, members of your department will be asked to fill out a questionnaire on a strictly voluntary and anonymous basis. The questionnaire should take no more than thirty minutes to complete and asks for information relating to the availability of ICT resources in your school and how these resources are used in the teacher's classroom practice. Members of your department may subsequently be invited to participate in the project as a case study teacher based on their answers to the questionnaire.

All information will be treated as strictly confidential*, coded references will be used in order to preserve the anonymity of individual teachers, your department and school. Your details will be held confidentially and will not be passed to any other organisation. The information given will be used for the purposes of this project, which may include publications.

There is no compulsion for you to participate in this study. If you do choose to participate, you are free to withdraw your department from the study until 1/2/2012, by contacting me using the contact details overleaf. Participation in the project will not disadvantage you in any way and there is no risk, discomfort or inconvenience (minor or major) involved.

The project adheres to the British Educational Research Association's Revised Ethical Guidelines for Educational Research (2004) http://www.bera.ac.uk/files/2008/09/ethica1.pdf and the King's College London Guidelines on Good Practice in Academic Research http://www.kcl.ac.uk/college/policyzone/index.php?id=247

Please read the information sheet and sign and return the attached consent form, along with any completed questionnaires, in the envelope provided.

If you have any questions about the project, please do not hesitate to contact me using the contact details overleaf. Thank you.

Yours sincerely

Nicola Bretscher
Post-graduate student
King's College London

Nicola Bretscher
Department of Education & Professional Studies
King's College London
Franklin-Wilkins Building (Waterloo Bridge Wing)
Waterloo Road
London SE1 9NH

Email: nicola.bretscher@kcl.ac.uk

Jeremy Hodgen (supervisor)
Department of Education & Professional Studies
King's College London
Franklin-Wilkins Building (Waterloo Bridge Wing)
Waterloo Road
London SE1 9NH

Tel: 020 7848 3102
Email: jeremy.hodgen@kcl.ac.uk

* Please note that there are exceptions to the obligation of confidentiality which would justify disclosure of confidential information. Any such disclosures would comply with the Data Protection Act and include:

- Public interest: There may be circumstances where the right to confidentiality must be weighed against the public interest where there is a real or serious risk that another, or the public at large, may be put in danger by the participant.

- Statutory Provisions: e.g. Children Act 1989 and 2004; Public Health (Control of Diseases) Act 1984 (notifiable diseases).

Consent form (participating institution)

Project: Using ICT to investigate Mathematical Knowledge in Teaching

I understand the nature and purpose of the project and agree to participate.

I agree that there is no compulsion for me or the members of my department to participate in this research and, if I do choose to participate, I may withdraw my participation before 1/2/2012 without penalty;

I understand that any information which I or the members of my department give will be used solely for the purposes of this research project, which may include publications. I understand that such information will be treated as strictly confidential and handled in accordance with the Data Protection Act 1998.

I understand that members of my department may subsequently be invited to participate as a case study teacher based on their answers to the questionnaire. I agree for them to be contacted in the future for this purpose.

…………………………………… ………………………
(*Signature, on behalf of institution*) (*Date*)

…………………………………………………………………………………………………
(Please print your name and official position in your institution and affix the institutional stamp below. Thank you.)

Please return the signed and stamped form to:

Nicola Bretscher
Department of Education & Professional Studies
King's College London
Franklin-Wilkins Building (Waterloo Bridge Wing)
Waterloo Road
London SE1 9NH
Email: nicola.bretscher@kcl.ac.uk

Thank you.

[date]

Dear [name of teacher]

Using ICT to investigate Mathematical Knowledge in Teaching

I am writing to invite you to participate as a case study teacher in my PhD research project at the Department of Education, King's College London. This project aims to investigate the mathematical knowledge that teachers draw upon in their teaching of mathematics using ICT. In this project ICT is also used as a tool to explore teachers' mathematical knowledge. You have been invited to be a case study teacher on the basis of your answers to the questionnaire you previously filled out on ICT use.

If you choose to participate, you will be asked to take part in three interviews lasting no more than one hour each and two observations of your teaching using ICT. The interviews will be recorded for transcription. One observation should be where ICT is used in a whole-class context; the other should be where students are given direct access to computers. The interviews and observations will happen at a time and place of your choosing. Participation in the study may make you more aware of how you use ICT resources in your teaching and may give you an opportunity to reflect on your professional practice. At the end of the study you will be able to receive information about the survey and case study results, should you wish.

Any information you give will be treated as strictly confidential* and pseudonyms will be used to preserve your anonymity. Your details will be held confidentially and will not be passed to any other organisation. The information you give will be used for the purposes of this project, which may include publications.

There is no compulsion for you to participate in this study. If you do choose to participate, you are free to withdraw from the study until 1/7/2012, by contacting me using the contact details overleaf. Participation in the project will not disadvantage you in any way and there is no risk, discomfort or inconvenience (minor or major) involved.

The project adheres to the British Educational Research Association's Revised Ethical Guidelines for Educational Research (2004) http://www.bera.ac.uk/files/2008/09/ethica1.pdf and the King's College London Guidelines on Good Practice in Academic Research http://www.kcl.ac.uk/college/policyzone/index.php?id=247

Please read the information sheet and sign and return the attached consent form.

If you have any questions about the project, please do not hesitate to contact me using the contact details overleaf. Thank you.

Yours sincerely

Nicola Bretscher
Post-graduate student
King's College London

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* Please note that there are exceptions to the obligation of confidentiality which would justify disclosure of confidential information. Any such disclosures would comply with the Data Protection Act and include:

- Public interest: There may be circumstances where the right to confidentiality must be weighed against the public interest where there is a real or serious risk that another, or the public at large, may be put in danger by the participant.

- Statutory Provisions: e.g. Children Act 1989 and 2004; Public Health (Control of Diseases) Act 1984 (notifiable diseases).

INFORMATION SHEET FOR CASE STUDY PARTICIPANTS

[redacted]

INFORMATION SHEET AND CONSENT FORM
Please complete this form after you have read this Information Sheet and before completing the questionnaire.

King's College Research Ethics Committee Ref:..

TITLE OF STUDY: Using ICT to investigate Mathematical Knowledge in Teaching

You are being invited to participate as a case study teacher in my PhD research project. Before you decide whether to take part, it is important for you to read the following information carefully and understand why the research is being done and what it will involve. You should only participate if you want to; choosing not to take part will not disadvantage you in any way. Please take time to read the following information carefully and discuss it with others if you wish. If you have any questions about the project, please do not hesitate to contact me using the contact details below.

What is the purpose of this study? This study aims to investigate the mathematical knowledge that teachers draw upon in their teaching of mathematics using ICT. In this project ICT is also used as a tool to explore teachers' mathematical knowledge.

Why have I been chosen to participate? You have been invited to be a case study teacher on the basis of your answers to the questionnaire you previously filled out on ICT use. Your school was chosen to take part in the survey due to it existing contacts with King's College.

What will happen to me if I take part? If you choose to participate, you will be asked to take part in three interviews lasting no more than one hour each and two observations of your teaching using ICT. One observation should be where ICT is used in a whole-class context; the other should be where students are given direct access to computers. The interviews and observations will happen at a time and place of your choosing.

What are the possible benefits? Participation in the study may make you more aware of how you use ICT resources in your teaching and may give you an opportunity to reflect on your professional practice. At the end of the study you will be able to receive information about the survey and case study results, should you wish.

Will my personal data be kept confidential? Your confidentiality will be ensured at all times and you will not be identified in any publication. Data from the study will be stored anonymously for seven years. You may withdraw your data from the project at any time, without giving any reason, before 1/7/2012.

Who is organising and funding the research? The research is part of the principal investigator's PhD research at King's College London and is funded by the ESRC. This study is reviewed by the College Research Ethics Committee of King's College London.

Contact for further information: The principal investigator of this study is Nicola Bretscher, Department of Education and Professional Studies. This is part of a PhD project, supervised by Dr. Jeremy Hodgen, Department of Professional and Educational Studies. Should you have any comments or concerns resulting from your participation in the study please contact me via email: nicola.bretscher@kcl.ac.uk
If this study has harmed you in any way you can contact King's College London using the details below for further advice and information: Jeremy Hodgen, email: jeremy.hodgen@kcl.ac.uk

[redacted]

CONSENT FORM FOR PARTICIPANTS IN RESEARCH STUDIES

[redacted]

King's College Research Ethics Committee
Ref:...

TITLE OF STUDY: Using ICT to investigate Mathematical Knowledge in Teaching

Thank you for considering taking part in this research. This form will be detached from your questionnaire and stored separately. Please complete the form:

Name of Teacher ....

I understand that there is no compulsion for me to participate in this research and, if I do choose to participate, I may at any stage withdraw my participation, without penalty, until 1/7/2012;

I understand that any information which I give will be used solely for the purposes of this research project, which may include publications;

I consent to the processing of my personal information for the purposes of this research study. I understand that such information will be treated as strictly confidential and handled in accordance with the Data Protection Act 1998.

…………………………………… ……………………

(*Signature of teacher*) (*Date*)

Appendix M. Additional statistical information

These tables provide additional statistical information to complement the results reported in Chapter 4, Section 4.3. The first four tables compare transmissionist measure across various background variables. The remaining tables in this appendix complement the results Tables in Section 4.3 as indicated. The results of two non-significant t-tests (see Age; Length of service) are reported here; the results of other t-tests are reported in the main body of the thesis.

Gender	**n**	**Mean (S.D.)**	**Mean diff (S.E.)**	**C.I. of difference Lower to Upper**	**Effect size**
Female	98	.064 (.515)	-.218 (.076)	-.368 to -.068	-.427
Male	84	.282 (.507)			

Age (years)	**n**	**Mean (S.D.)**	**Mean diff (S.E.)**	**C.I. of difference Lower to Upper**	**Effect size**
30 or under	59	.166 (.429)	.005 (.076)	-.144 to .155	.011
Over 30	122	.161 (.564)			

No significant difference: df = 146.4, t = .069, p = .945

Length of service (years)	**n**	**Mean (S.D.)**	**Mean diff (S.E.)**	**C.I. of difference Lower to Upper**	**Effect size**
6 or under	89	.191 (.473)	.052 (.077)	-.101 to .205	.099
Over 6 years	93	.139 (.565)			

No significant difference: df = 180, t = .670, p = .504

Frequency of ICT use	**n**	**Mean (S.D.)**	**Mean diff (S.E.)**	**C.I. of difference Lower to Upper**	**Effect size**
Frequent users	61	.063 (.548)	-.152 (.082)	-.313 to .009	-.294
Occasional users	118	.216 (.502)			

IWB: Frequency of software use		n	Mean (S.D.)	Mean diff (S.E.)	Effect size
CD-Roms	freq	36	.089 (.492)	-.100 (.099)	-.189
	occ	139	.189 (.539)		
Database	freq	23	.089 (.442)	-.088 (.119)	-.165
	occ	151	.176 (.543)		
Email	freq	53	.121 (.454)	-.086 (.087)	-.164
	occ	119	.207 (.555)		
Graphing software	freq	49	.166 (.553)	-.017 (.089)	-.031
	occ	122	.182 (.517)		
Dynamic geometry	freq	30	-.099 (.570)	-.324 (.104)	-.628
	occ	145	.225 (.505)		
IWB software	freq	146	.115 (.516)	-.285 (.100)	-.551
	occ	33	.340 (.515)		
Logo	freq	15	-.006 (.590)	-.196 (.144)	-.369
	occ	151	.190 (.527)		
MyMaths	freq	116	.069 (.481)	-.284 (.079)	-.557
	occ	64	.353 (.558)		
Other websites	freq	112	.065 (.500)	-.315 (.080)	-.624
	occ	61	.380 (.516)		
PowerPoint	freq	107	.071 (.495)	-.251 (.078)	-.489
	occ	72	.322 (.540)		
SMILE	freq	11	.068 (.500)	-.130 (.169)	-.240
	occ	152	.198 (.542)		
Spreadsheet	freq	45	.039 (.498)	-.174 (.090)	-.331
	occ	134	.212 (.532)		
Word	freq	74	.101 (.480)	-.119 (.080)	-.227
	occ	105	.220 (.556)		

See also Table 4.2

Computer suite: Frequency of software use		n	Mean (S.D.)	Mean diff (S.E.)	Effect size
CD-Roms	freq	33	.081 (.503)	-.129 (.102)	-.245
	occ	136	.210 (.533)		
Database	freq	29	.163 (.508)	-.013 (.109)	-.025
	occ	143	.177 (.540)		
Email	freq	45	.016 (.483)	-.230 (.091)	-.442
	occ	123	.248 (.535)		
Graphing software	freq	88	.048 (.544)	-.240 (.079)	-.302
	occ	85	.288 (.496)		
Dynamic geometry	freq	83	-.004 (.521)	-.323 (.077)	-.638
	occ	90	.319 (.493)		
IWB software	freq	69	.072 (.479)	-.158 (.081)	-.302
	occ	102	.230 (.550)		
Logo	freq	19	-.120 (.541)	-.330 (.127)	-.633
	occ	152	.210 (.518)		
MyMaths	freq	129	.109 (.530)	-.233 (.092)	-.444
	occ	44	.342 (.512)		
Other websites	freq	126	.082 (.516)	-.328 (.087)	-.642
	occ	48	.410 (.497)		
PowerPoint	freq	90	.115 (.473)	-.123 (.082)	-.230
	occ	82	.239 (.588)		
SMILE	freq	15	-.244 (.541)	-.461 (.142)	-.878
	occ	152	.217 (.523)		
Spreadsheet	freq	103	.076 (.502)	-.227 (.080)	-.435
	occ	72	.302 (.547)		
Word	freq	89	.069 (.478)	-.216 (.079)	-.413
	occ	84	.285 (.566)		

See also Table 4.3

Frequency of hardware use		**n**	**Mean (S.D.)**
IWB	freq	158	.127 (.508)
	occ	25	.427 (.546)
Data projector	freq	67	.111 (.530)
	occ	114	.200 (.521)
Computer suite (shared)	freq	32	.008 (.494)
	occ	150	.207 (.522)
Computer suite (maths only)	freq	10	-.003 (.490)
	occ	170	.170 (.523)
Laptops	freq	13	.214 (.554)
	occ	170	.164 (.521)
Graphic calculator	freq	32	-.054 (.523)
	occ	149	.212 (.513)

See also Table 4.4

IWB: ICT orientation		**n**	**Mean (S.D.)**
confident using ICT	pos	152	.139 (.515)
	neg	24	.200 (.463)
More time needed for prep	pos	52	.107 (.503)
	neg	126	.189 (.520)
ICT contributes to learning	pos	135	.101 (.512)
	neg	45	.334 (.515)
ICT improves engagement	pos	147	.129 (.506)
	neg	34	.298 (.563)
Students' lack of familiarity	pos	80	.167 (.508)
	neg	101	.155 (.532)
ICT helps understanding	pos	140	.105 (.518)
	neg	40	.336 (.486)
Management more difficult	pos	149	.110 (.521)
	neg	31	.388 (.453)
Cover more ground	pos	99	.117 (.506)
	neg	81	.227 (.521)

See also Table 4.5

Computer Suite: ICT orientation		n	Mean (S.D.)
confident using ICT	pos	140	.124 (.502)
	neg	32	.378 (.610)
More time needed for prep	pos	51	.223 (.471)
	neg	122	.144 (.554)
ICT contributes to learning	pos	129	.115 (.530)
	neg	44	.334 (.513)
ICT improves engagement	pos	135	.140 (.524)
	neg	38	.267 (.550)
Students' lack of familiarity	pos	73	.144 (.553)
	neg	99	.189 (.522)
ICT helps understanding	pos	127	.113 (.513)
	neg	45	.309 (.558)
Management more difficult	pos	81	.103 (.553)
	neg	92	.223 (.505)
Cover more ground	pos	36	-.018 (.443)
	neg	135	.214 (.545)

See also Table 4.6

IWB practices		n	Mean (S.D.)
Teacher presentation	freq	138	.167 (.489)
	occ	40	.129 (.619)
Student discussion	freq	78	-.017 (.527)
	occ	102	.295 (.477)
Teacher control	freq	146	.170 (.488)
	occ	33	.110 (.660)
Explore students' ideas	freq	50	-.021 (.540)
	occ	130	.229 (.499)
Prevent discrepanies	freq	74	.149 (.536)
	occ	96	.162 (.514)
Students control	freq	39	-.105 (516)
	occ	141	.233 (.500)
Highlight discrepancies	freq	55	.056 (.536)
	occ	118	.208 (.512)
Avoid mistakes	freq	77	.236 (.505)
	occ	99	.101 (.535)

See also Table 4.7

Appendix N. Back cover material

Included as separate sheets: Figure 6.8, Figure 6.9, Figure 7.1

Included on CD: Robert's GeoGebra interview, mins 5.05 – 12.10, and the GeoGebra interview file on circle theorems